

Differential 1-forms on diffeological spaces and diffeological gluing

Ekaterina Pervova

December 20, 2016

Abstract

This paper aims to describe the behavior of diffeological differential 1-forms under the operation of gluing of diffeological spaces along a smooth map. In the diffeological context, two constructions regarding diffeological forms are available, that of the vector space $\Omega^1(X)$ of all 1-forms, and that of the pseudo-bundle $\Lambda^1(X)$ of values of 1-forms. We describe the behavior of the former under an arbitrary gluing of two diffeological spaces, while for the latter, we limit ourselves to the case of gluing along a diffeomorphism.

MSC (2010): 53C15 (primary), 57R35, 57R45 (secondary).

Introduction

The aim of this work is rather modest; it is to examine the behavior of diffeological differential forms (1-forms, usually, but a lot of it naturally holds for forms of higher order) under the operation of diffeological gluing. In fact, assuming that the notation is known, the main question can be stated very simply: *if a diffeological space X_1 is glued to a diffeological space X_2 along a map f , how can we obtain the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ out of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$?* One aim in answering it is to consider the behavior under gluing of *diffeological connections*, defined as operators $C^\infty(X, V) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V)$, where $\pi : V \rightarrow X$ is a diffeological vector pseudo-bundle and $C^\infty(X, V)$ is the space of smooth sections of it (as this is done in [9]).

Of course, we make no assumption, as to any of these symbols or terms being known (although the explanation of them can be found in the excellent book [4]), so here we give a rough description of their meaning, and give precise definitions of the most important ones in the first two sections. A *diffeological space* is a set equipped with a *diffeology*, a set of maps into it that are declared to be smooth. There are ensuing notions of smooth maps between such spaces, the induced diffeologies of all kinds, among which we mention in particular the *subset diffeology* and the *quotient diffeology*, for the simple reason that they provide for any subset, and any quotient, of a diffeological space, being in turn a diffeological space, in striking contrast with the category of smooth manifolds.

This latter property makes for the operation of diffeological gluing to be well-defined in the diffeological context. In essence, we are talking about the notion of topological gluing: given two sets (say, they are topological spaces) X_1 and X_2 and a map $f : X_1 \supset Y \rightarrow X_2$, the usual gluing procedure yields the space $(X_1 \sqcup X_2)_{/x_2=f(x_1)}$, which for a continuous f has a natural topology. Now, the just-mentioned property of diffeology ensures the same thing, if we assume that f is smooth as a map on Y , which inherits its diffeology from X_1 .

There is a certain correlation between this operation being well-defined in the diffeological setting, and the fact that the diffeological counterpart of a vector bundle is a *diffeological vector pseudo-bundle*, and in general it is not a bundle at all. The reason for it not being a bundle, not in the usual sense, is simply that it is allowed to have fibres of different dimensions (which are still required to be vector spaces with smooth operations), and the necessity of such objects for diffeology is not just aprioristic; they arise naturally in various aspects of the theory (see Example 4.13 of [2] for an instance of this). The aforementioned correlation with the operation of gluing is that such pseudo-bundles, when they are not too intricate, can frequently be obtained by applying diffeological gluing to a collection of usual smooth vector bundles.

Diffeological vector spaces and particularly diffeological pseudo-bundles give the appropriate framework for differential forms on diffeological spaces. By itself, a differential k -form on a diffeological space

X is just a collection of usual k -forms, one for each plot and defined on the domain of the definition of the plot; a very natural smooth compatibility is imposed on this collection to ensure consistency with (usual) smooth substitutions on the domains of plots. The collection of all possible k -forms defined in this fashion is naturally a diffeological vector space and is denoted by $\Omega^1(X)$, but it does not fiber naturally over X ; a further construction, a certain space $\Lambda^1(X)$, has a pseudo-bundle structure, and this is our main object of study.

The main results These regard three main points: the diffeological vector space $\Omega^1(X_1 \cup_f X_2)$, the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$, and construction of so-called pseudo-metrics on the latter. As for the former, our main result is as follows.

Theorem 1. *Let X_1 and X_2 be two diffeological spaces, let $f : X_1 \supset Y \rightarrow X_2$ be a smooth map, and let $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$ be the natural inclusions. Then $\Omega^1(X_1 \cup_f X_2)$ is diffeomorphic to the subset of $\Omega^1(X_1) \times \Omega^1(X_2)$ consisting of all pairs (ω_1, ω_2) such that $i^*\omega_1 = f^*j^*\omega_2$.*

The description is much less straightforward when it comes to the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$, and we only give it in the case where f is a diffeological diffeomorphism. Even in that case, it is easier to say what it is not rather than what it is. We indicate here that, with some conditions on the gluing map f , each fibre of $\Lambda^1(X_1 \cup_f X_2)$ coincides either with a fibre of $\Lambda^1(X_1)$, or one of $\Lambda^1(X_2)$ (more generally, with a subspace of one of them), or with a subset of the direct product of the two. Accordingly, $\Lambda^1(X_1 \cup_f X_2)$ is equipped with two standard projections, each of them defined on a proper subset of it, to $\Lambda^1(X_1)$ one, to $\Lambda^1(X_2)$ the other. Under a certain technical condition, that we write as $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ (see Section 7.3.1 for the definition) and that is satisfied, for example, for affine subspaces of the standard \mathbb{R}^n , the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ can be characterized as the coarsest one for which these two projections are smooth:

Theorem 2. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism of its domain with its image such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Let $\pi^\Lambda : \Lambda^1(X_1 \cup_f X_2) \rightarrow X_1 \cup_f X_2$, $\pi_1^\Lambda : \Lambda^1(X_1) \rightarrow X_1$, and $\pi_2^\Lambda : \Lambda^1(X_2) \rightarrow X_2$ be the pseudo-bundle projections. Then*

$$\Lambda^1(X_1 \cup_f X_2) \cong \cup_{x_1 \in X_1 \setminus Y} \Lambda_{x_1}^1(X_1) \bigcup \cup_{y \in Y} \left(\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2) \right) \bigcup \cup_{x_2 \in X_2 \setminus f(Y)} \Lambda_{x_2}^1(X_2),$$

where \cong has the following meaning:

- the set $\cup_{x_1 \in X_1 \setminus Y} \Lambda_{x_1}^1(X_1) \subset \Lambda^1(X)$ is identified with $(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))$ and with $(\pi_1^\Lambda)^{-1}(X_1 \setminus Y)$. This identification is a diffeomorphism for the subset diffeologies relative to the inclusions

$$(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y)) \subset \Lambda^1(X_1 \cup_f X_2) \quad \text{and} \quad (\pi_1^\Lambda)^{-1}(X_1 \setminus Y) \subset \Lambda^1(X_1);$$

- likewise, the set $\cup_{x_2 \in X_2 \setminus f(Y)} \Lambda_{x_2}^1(X_2)$ is identified with $(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))$ and with $(\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y))$, with the identification being again a diffeomorphism for the subset diffeologies relative to the inclusions

$$(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y))) \subset \Lambda^1(X_1 \cup_f X_2) \quad \text{and} \quad (\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y));$$

- finally, the set $\cup_{y \in Y} \left(\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2) \right)$ is given the direct sum diffeology as the appropriate (determined by compatibility) subset of the result of the direct of the following two restricted pseudo-bundles:

$$\pi_1^\Lambda|_{(\pi_1^\Lambda)^{-1}(Y)} : (\pi_1^\Lambda)^{-1}(Y) \rightarrow Y \quad \text{and} \quad f^{-1} \circ \pi_2^\Lambda|_{(\pi_2^\Lambda)^{-1}(f(Y))} : (\pi_2^\Lambda)^{-1}(f(Y)) \rightarrow Y.$$

This direct sum subset can also be identified with $(\pi^\Lambda)^{-1}(i_2(f(Y)))$ and given the subset diffeology relative to the inclusion

$$(\pi^\Lambda)^{-1}(i_2(f(Y))) \subset \Lambda^1(X_1 \cup_f X_2);$$

once again, this identification is a diffeomorphism for the above direct sum diffeology and the just-mentioned subset diffeology.

A more precise form of this statement is Theorem 8.13. As an application of it, we consider (under appropriate assumptions) a construction of a *pseudo-metric* on $\Lambda^1(X_1 \cup_f X_2)$, which is a counterpart of a Riemannian metric for finite-dimensional diffeological vector pseudo-bundles.

Acknowledgments The scope of this work is very much limited, but nonetheless carrying it out required a degree of patience and good humor. I may or may not have a natural propensity to these, but in any case it certainly helped to have a good example, for which I must most convincingly thank Prof. Riccardo Zucchi.

1 Main definitions

We recall here as briefly as possible the basic definitions (and some facts) regarding diffeological spaces, diffeological pseudo-bundles, and diffeological gluing; the definitions regarding differential forms are collected in the section that follows.

1.1 Diffeological spaces

The notion of a **diffeological space** is due to J.M. Souriau [10], [11]; it is defined as a (n arbitrary) set X equipped with a **diffeology**. A diffeology, or a diffeological structure, on X is a set \mathcal{D} of maps $U \rightarrow X$, where U is any domain in \mathbb{R}^n (and, for a fixed X , this n might vary); the set \mathcal{D} must possess the following properties. First, it must include all constant maps into X ; second, for any $p \in \mathcal{D}$ its pre-composition $p \circ g$ with any usual smooth map g must again belong to \mathcal{D} ; and third, if $p : U \rightarrow X$ is a set map and U admits an open cover by some sub-domains U_i such that $p|_{U_i} \in \mathcal{D}$ then necessarily $p \in \mathcal{D}$. The maps that compose a given diffeology \mathcal{D} on X are called **plots** of \mathcal{D} (or of X).

Finer and coarser diffeologies on a given set On a fixed set X , there can be many diffeologies; and these being essentially sets of maps, it makes sense (in some cases) to speak of one being included in another;¹ the former is then said to be **finer** and the latter, to be **coarser**. It is particularly useful, on various occasions, to consider the finest (or the coarsest) diffeology possessing a given property P ; many definitions are stated in such terms (although the diffeology thus defined can, and usually is, also be given an explicit description).

Smooth maps, pushforwards, and pullbacks Given two diffeological spaces X and Y , a set map $f : X \rightarrow Y$ is said to be **smooth** if for any plot p of X the composition $f \circ p$ is a plot of Y . The *vice versa* (that is, that every plot of Y admits, at least locally, such a form for some p) does not have to be true, but if it is, one says that the diffeology of Y is the **pushforward** of the one of X by the map f ; or, accordingly, that the diffeology of X is the **pullback** of that of Y .

Topological constructions and diffeologies Given one or more (as appropriate) diffeological spaces, there are standard diffeological counterparts of all the basic set-theoretic and topological constructions, such as taking subspaces, quotients, direct products, and disjoint unions (with more complicated constructions following automatically). What we mean by a standard diffeological counterpart is of course the choice of diffeology, since the underlying set is known. Thus, any subset X' of a diffeological space X has the standard diffeology that is called the **subset diffeology** and that consists of precisely those of plots of X whose range is contained in X' ; the quotient of X by any equivalence relation \sim has the **quotient diffeology** that is the pushforward of the diffeology of X by the quotient projection $X \rightarrow X/\sim$. The direct product of a collection of diffeological spaces carries the **direct product diffeology** that is the coarsest diffeology such that all projections on individual factors are smooth; and the disjoint union, the **disjoint union diffeology**, defined as the finest diffeology such that the inclusion of each component is a smooth map.

¹Formally speaking, the diffeologies on any given set are partially ordered with respect to inclusion and form a complete lattice; see Chapter 1 of [4] for more details.

Diffeologies on spaces of functions For any pair X and Y of diffeological spaces, we can consider the space $C^\infty(X, Y)$ of all smooth (in the diffeological sense) maps $X \rightarrow Y$. This space is also endowed with its standard diffeology that is called the **functional diffeology** and that can be defined as follows. A map $q : U \rightarrow C^\infty(X, Y)$ is a plot for this functional diffeology if and only if for every plot $p : U' \rightarrow X$ of X the natural evaluation map $U \times U' \ni (u, u') \rightarrow q(u)(p(u')) \in Y$ is a plot of Y .

1.2 Diffeological vector pseudo-bundles

We briefly mention this concept, since we will need it in order to consider $\Lambda^1(X)$ (see Introduction and the following Section). A smooth surjective map $\pi : V \rightarrow X$ between two diffeological spaces V and X is a **diffeological vector pseudo-bundle** if for all $x \in X$ the pre-image $\pi^{-1}(x)$ carries a vector space structure, and the corresponding addition $V \times_X V \rightarrow V$ and scalar multiplication $\mathbb{R} \times V \rightarrow V$ maps are smooth (for the natural diffeologies on their domains). This is a diffeological counterpart of the usual smooth vector bundle; we stress however that it does not include the requirement of local triviality. Indeed, various examples that motivated the concept do not enjoy this property, although there are contexts in which it is necessary to add the assumption of it.²

Diffeological vector spaces and operations on them Each fibre of a diffeological vector pseudo-bundle is a vector space and a diffeological space at the same time; and the operations are actually smooth maps for the subset diffeology. Thus, the fibres are **diffeological vector spaces** (that are defined as vector spaces endowed with a diffeology for which the addition and scalar multiplication maps are smooth). We briefly mention that all the basic operations on vector spaces (subspaces, quotients, direct sums, tensor products, and duals) have their diffeological counterparts (see [13], [15]), in the sense of there being a standard choice of diffeology on the resulting vector space. Thus, a subspace is endowed with the subset diffeology, the quotient space, with a quotient one, the direct sum carries the product diffeology, and the tensor product, the quotient diffeology relative to the product diffeology on the (free) product of its factors.

The case of the dual spaces is worth mentioning in a bit more detail, mainly because there usually is not the standard isomorphism by duality between V and V^* , not even for finite-dimensional V . Indeed, the diffeological dual V^* is defined as $C^\infty(V, \mathbb{R})$, where \mathbb{R} has standard diffeology, and, unless V is also standard, V^* has smaller dimension than V . The diffeology on V^* is the functional diffeology (see above). Notice also that if V is finite-dimensional, V^* is always a standard space (see [6]).

Operations of diffeological vector pseudo-bundles The usual operations on vector bundles (direct sum, tensor product, dual bundle) are defined for diffeological vector pseudo-bundles as well (see [13]), although in the absence of local trivializations defining them does not follow the standard strategy. Indeed, they are defined by carrying out these same operations fibrewise (which is still standard), but then are endowed with a diffeology that either described explicitly, or characterized as the finest diffeology inducing the already-existing diffeology on each fibre. For instance, if $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ are two finite-dimensional diffeological vector pseudo-bundles over the same base space X , their direct sum $\pi_1 \oplus \pi_2 : V_1 \oplus V_2 \rightarrow X$ is defined by setting $V_1 \oplus V_2 := \cup_{x \in X} (\pi_1^{-1}(x) \oplus \pi_2^{-1}(x))$ and endowing it with the finest diffeology such that the corresponding subset diffeology on each fibre $\pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$ is its usual direct sum diffeology (see [13]). We will not make much use of most of these operations and so do not go into more detail about them (see [13]; also [7] and [8] for more details), mentioning the only property that we will need in the sequel and that regards sub-bundles.

Let $\pi : V \rightarrow X$ be a diffeological vector pseudo-bundle. For each $x \in X$ let $W_x \leq \pi^{-1}(x)$ be a vector subspace, and let $W = \cup_{x \in X} W_x$. It is endowed with the obvious projection onto X , and as a subset of V , it carries the subset diffeology (which on each fibre W_x induces the same diffeology as that relative to the inclusion $W_x \leq \pi^{-1}(x)$). This diffeology makes W into a diffeological vector pseudo-bundle and is said to be a **sub-bundle** of V ; we stress that there are no further conditions on the choice of W_x , as long as each of them is a vector subspace in the corresponding fibre.

²These contexts mostly have to do with attempts to endow these pseudo-bundles with a kind of pseudo-Riemannian structure; we will deal very little with these in the present paper.

Pseudo-metrics It is known (see [4]) that a finite-dimensional diffeological vector space admits a smooth scalar product if and only if it is a standard space; in general, the closest that comes to a scalar product on such a space is a smooth symmetric semi-definite positive bilinear form of rank equal to the dimension of the diffeological dual (see [6]). Such a form is called a **pseudo-metric** on the space in question.

It is then obvious that neither a diffeological vector pseudo-bundle would usually admit a diffeologically smooth Riemannian metric (it would have to have all standard fibres, and this condition is still not sufficient). However, it may admit the extension of the notion of pseudo-metric (called pseudo-metric as well), which is just a section of the tensor square of the dual pseudo-bundle such that its value at each point is a pseudo-metric, in the sense of diffeological vector spaces, on the corresponding fibre. The precise definition is as follows.

Let $\pi : V \rightarrow X$ be a finite-dimensional diffeological vector pseudo-bundle. A **pseudo-metric** on it is a smooth section $g : X \rightarrow V^* \otimes V^*$ such that for all $x \in X$ the value $g(x)$ is a smooth symmetric semidefinite-positive bilinear form on $\pi^{-1}(x)$ of rank equal to $\dim((\pi^{-1}(x))^*)$ (see [7] or [8] for more details).

1.3 Diffeological gluing

This concept, which is central to the present paper, is just a natural carry-over of the usual topological gluing to the diffeological context.

1.3.1 Gluing of diffeological spaces and maps between them

Gluing together two diffeological spaces along a map between subsets of them is the main building block of this construction. It then naturally extends to define a gluing of smooth maps, and in particular (also a central concept for us) of diffeological pseudo-bundles.

Diffeological spaces Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supset Y \rightarrow X_2$ be a smooth (for the subset diffeology on Y) map. The result of (diffeological) gluing of X_1 to X_2 along f is the space $X_1 \cup_f X_2$ defined by

$$X_1 \cup_f X_2 = (X_1 \sqcup X_2) / \sim,$$

where \sim is the equivalence relation determined by f , that is, $Y \ni y \sim f(y)$. The diffeology on $X_1 \cup_f X_2$, called the **gluing diffeology**, is the pushforward of the disjoint union diffeology on $X_1 \sqcup X_2$ by the quotient projection $\pi : X_1 \sqcup X_2 \rightarrow X_1 \cup_f X_2$. Since a pushforward diffeology (equivalently, quotient diffeology) is the finest one making the defining projection smooth, it is quite obvious that the gluing diffeology is the finest one induced³ by the diffeologies on its factors. Indeed, it frequently turns out to be weaker than other natural diffeologies that the resulting space might carry, as it occurs for the union of the coordinate axes in \mathbb{R}^2 , whose gluing diffeology (relative to gluing of the two standard axes at the origin) is finer than the subset diffeology relative to its inclusion in \mathbb{R}^2 , see Example 2.67 in [14].

We remark that we only really consider the case of gluing of two diffeological spaces. However,⁴ it can easily be extended to a finite sequence of gluings.

The standard disjoint cover of $X_1 \cup_f X_2$ There is a technical convention, which comes in handy when working with glued spaces, for instance, when defining maps on them (see below for an instance of this). It is based on the trivial observation that the following two maps are inductions,⁵ and their ranges form a disjoint cover of $X_1 \cup_f X_2$:

$$i_1^{X_1} : X_1 \setminus Y \hookrightarrow (X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2 \text{ and } i_2^{X_2} : X_2 \hookrightarrow (X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2,$$

where in both cases the second arrow is the quotient projection π . We will omit the upper index when it is clear which glued space we are referring to.

³We use the term informally at the moment; it stands for whatever diffeology can be obtained in not-too-artificial a way from those on the factors.

⁴As one of the referees of the previous version of this paper pointed out.

⁵An injective map $f : X \rightarrow Y$ between two diffeological spaces is called an induction if the diffeology of X is the pullback of the subset diffeology on $f(X) \subset Y$.

Smooth maps Let us now have two pairs of diffeological spaces, X_1, X_2 and Z_1, Z_2 , with a gluing within each pair given respectively by $f : X_1 \supseteq Y \rightarrow X_2$ and $g : Z_1 \supseteq Y' \rightarrow Z_2$. Then, under a specific condition called **(f, g) -compatibility**, two maps $\varphi_i \in C^\infty(X_i, Z_i)$ for $i = 1, 2$ induce a well-defined map in $C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$.

The (f, g) -compatibility means that $\varphi_1(Y) = Y'$ and $g \circ \varphi_1|_Y = \varphi_2 \circ f$. The induced map, denoted by $\varphi_1 \cup_{(f,g)} \varphi_2$, is given by

$$(\varphi_1 \cup_{(f,g)} \varphi_2)(x) = \begin{cases} i_1^{Z_1}(\varphi_1((i_1^{X_1})^{-1}(x))) & \text{if } x \in \text{Range}(i_1^{X_1}) \\ i_2^{Z_2}(\varphi_2((i_2^{X_2})^{-1}(x))) & \text{if } x \in \text{Range}(i_2^{X_2}). \end{cases}$$

Furthermore, the assignment $(\varphi_1, \varphi_2) \mapsto \varphi_1 \cup_{(f,g)} \varphi_2$ defines a map $C^\infty(X_1, Z_1) \times_{\text{comp}} C^\infty(X_2, Z_2) \rightarrow C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$ from the set of all (f, g) -compatible pairs (φ_1, φ_2) to $C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$. This map is smooth for the functional diffeology on the latter space and for the subset diffeology (relative to the product diffeology on the ambient space $C^\infty(X_1, Z_1) \times C^\infty(X_2, Z_2)$) on its domain of definition $C^\infty(X_1, Z_1) \times_{\text{comp}} C^\infty(X_2, Z_2)$ (see [8]).

1.3.2 Gluing of pseudo-bundles

Gluing of two diffeological vector pseudo-bundles is an operation which is essentially a special case of gluing of two smooth maps (see immediately above). Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let $f : X_1 \supseteq Y \rightarrow X_2$ be a smooth map defined on some subset of X_1 , and let $\tilde{f} : \pi_1^{-1}(Y) \rightarrow \pi_2^{-1}(f(Y))$ be a smooth lift of f whose restriction to each fibre in $\pi_1^{-1}(Y)$ is linear. The definitions given so far allow us to consider (without any further comment) the spaces $V_1 \cup_{\tilde{f}} V_2$ and $X_1 \cup_f X_2$, and the map $\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2$ between them. It then follows from the assumptions on \tilde{f} that this latter map is, in turn, a diffeological vector pseudo-bundle, with operations on fibres inherited from either V_1 or V_2 , as appropriate (see [7]).

This gluing operation is relatively well-behaved with respect to the usual operations on smooth vector bundles, which, as we mentioned above, extend to the diffeological pseudo-bundles. More precisely, it commutes with the direct sum and tensor product, but in general not with taking dual pseudo-bundles. We do not give more details about these, since we will not need them.

2 Diffeological differential 1-forms

For diffeological spaces, there exists a rather well-developed theory of differential k -forms on them (see [4], Chapter 6, for a detailed exposition). We now recall the case $k = 1$ (some definitions are given also for generic k).

2.1 Differential 1-forms and differentials of functions

A **diffeological differential 1-form** on a diffeological space X is defined by assigning to each plot $p : \mathbb{R}^k \supset U \rightarrow X$ a usual differential 1-form $\omega(p)(u) = f_1(u)du_1 + \dots + f_k(u)du_k \in \Lambda^1(U)$ such that this assignment satisfies the following compatibility condition: if $q : U' \rightarrow X$ is another plot of X such that there exists a usual smooth map $F : U' \rightarrow U$ with $q = p \circ F$ then $\omega(q)(u') = F^*(\omega(p)(u))$.

Let now $f : X \rightarrow \mathbb{R}$ be a diffeologically smooth function on it; recall that this means that for every plot $p : U \rightarrow X$ the composition $f \circ p : U \rightarrow \mathbb{R}$ is smooth in the usual sense, therefore $d(f \circ p)$ is a differential form on U . It is quite easy to see that the assignment $p \mapsto d(f \circ p) =: \omega_p$ is a differential 1-form on X ; this is called the **differential** of f . To see that it is well-defined, let $g : V \rightarrow U$ be a smooth function. The smooth compatibility condition $\omega_{p \circ g} = g^*(\omega_p)$ is then equivalent to $d((f \circ p) \circ g) = g^*(d(f \circ p))$, a standard property of differential forms in the usual sense.

2.2 The space $\Omega^1(X)$ of 1-forms

The set of all differential 1-forms on X is denoted by $\Omega^1(X)$; it carries a natural functional diffeology with respect to which it becomes a diffeological vector space. There is also a (pointwise) quotient of it

over the forms degenerating at the given point; the collection of such quotients forms a (pseudo-)bundle $\Lambda^1(X)$.

The functional diffeology on $\Omega^1(X)$ The addition and the scalar multiplication operations, that make $\Omega^1(X)$ into a vector space, are given pointwise (meaning the points in the domains of plots). The already-mentioned functional diffeology on $\Omega^1(X)$ is characterized by the following condition:

- a map $q : U' \rightarrow \Omega^1(X)$ is a plot of $\Omega^1(X)$ if and only if for every plot $p : U \rightarrow X$ the map $U' \times U \rightarrow \Lambda^1(\mathbb{R}^n)$ given by $(u', u) \mapsto q(u')(p)(u)$ is smooth, where $U \subset \mathbb{R}^n$.

The expression $q(u')(p)$ stands for the 1-form on the domain of definition of p , *i.e.*, the domain U , that the differential 1-form $q(u')$ on X assigns to the plot p of X .

2.3 The bundle of k -forms $\Lambda^k(X)$

Once again, our main interest here is the case of $k = 1$; we treat the general case simply because it does not change much.

The fibre $\Lambda_x^k(X)$ There is a natural quotienting of $\Omega^k(X)$, which gives, at every point $x \in X$, the set of all distinct values, at x , of the differential k -forms on X . This set is called $\Lambda_x^k(X)$; its precise definition is as follows.

Let X be a diffeological space, and let x be a point of it. A plot $p : U \rightarrow X$ is *centered at x* if $U \ni 0$ and $p(0) = x$. Let \sim_x be the following equivalence relation: two k -forms $\alpha, \beta \in \Omega^k(X)$ are equivalent, $\alpha \sim_x \beta$, if and only if, for every plot p centered at x , we have $\alpha(p)(0) = \beta(p)(0)$. The class of α for the equivalence relation \sim_x is called **the value of α at the point x** and is denoted by α_x . The set of all the values at the point x , for all k -forms on X , is denoted by $\Lambda_x^k(X)$:

$$\Lambda_x^k(X) = \Omega^k(X) / \sim_x = \{\alpha_x \mid \alpha \in \Omega^k(X)\}.$$

The space $\Lambda_x^k(X)$ is called the **space of k -forms of X at the point x** .

The space $\Lambda_x^k(X)$ as a quotient of $\Omega^k(X)$ Two k -forms α and β have the same value at the point x if and only if their difference vanishes at this point: $(\alpha - \beta)_x = 0$. The set $\{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}$ of the k -forms of X vanishing at the point x is a vector subspace of $\Omega^k(X)$; furthermore,

$$\Lambda_x^k(X) = \Omega^k(X) / \{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}.$$

In particular, as a quotient of a diffeological vector space by a vector subspace, the space $\Lambda_x^k(X)$ is naturally a diffeological vector space; the addition and the scalar multiplication on $\Lambda_x^k(X)$ are well-defined for any choice of representatives.

The k -forms bundle $\Lambda^k(X)$ The **bundle of k -forms over X** , denoted by $\Lambda^k(X)$, is the union of all spaces $\Lambda_x^k(X)$:

$$\Lambda^k(X) = \coprod_{x \in X} \Lambda_x^k(X) = \{(x, \alpha) \mid \alpha \in \Lambda_x^k(X)\}.$$

It has the obvious structure of a pseudo-bundle over X . The bundle $\Lambda^k(X)$ is endowed with the diffeology that is the pushforward of the product diffeology on $X \times \Omega^k(X)$ by the projection $\Pi : X \times \Omega^k(X) \rightarrow \Lambda^k(X)$ acting by $\Pi(x, \alpha) = (x, \alpha_x)$. Note that for this diffeology the natural projection $\pi : \Lambda^k(X) \rightarrow X$ is a local subduction;⁶ furthermore, each subspace $\pi^{-1}(x)$ is smoothly isomorphic to $\Lambda_x^k(X)$.

⁶A surjective map $f : X \rightarrow Y$ between two diffeological spaces is called a subduction if the diffeology of Y coincides with the pushforward of the diffeology of X by f .

The plots of the bundle $\Lambda^k(X)$ A map $p : U \ni u \mapsto (p_1(u), p_2(u)) \in \Lambda^k(X)$ defined on some domain U in some \mathbb{R}^m is a plot of $\Lambda^k(X)$ if and only if the following two conditions are fulfilled:

1. The map p_1 is a plot of X ;
2. For all $u \in U$ there exists an open neighborhood U' of u and a plot $q : U' \rightarrow \Omega^k(X)$ (recall that $\Omega^k(X)$ is considered with its functional diffeology described above) such that for all $u' \in U'$ we have $p_2(u') = q(u')(p_1(u'))$.

In other words, a plot of $\Lambda^k(X)$ is locally represented by a pair, consisting of a plot of X and a plot of $\Omega^k(X)$ (with the same domain of definition).

3 The spaces $\Omega^1(X_1 \cup_f X_2)$, $\Omega^1(X_1 \sqcup X_2)$, and $\Omega^1(X_1) \times \Omega^1(X_2)$

Let X_1 and X_2 be two diffeological spaces, and $f : X_1 \supset Y \rightarrow X_2$ is a smooth map that defines a gluing between them. We now describe how the space $\Omega^1(X_1 \cup_f X_2)$ is related to the spaces $\Omega^1(X_1)$ and $\Omega^1(X_2)$.

Since the space $X_1 \cup_f X_2$ is a quotient of the disjoint union $X_1 \sqcup X_2$, the natural projection $\pi : (X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2$ yields the corresponding pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1 \sqcup X_2)$ (see [4], Section 6.38); as we show immediately below, the latter space is diffeomorphic to $\Omega^1(X_1) \times \Omega^1(X_2)$. We then consider the image of π^* (this space is sometimes called the space of *basic* forms); we show that, although in general π^* is not surjective, it is a diffeomorphism with its image. Finally, we describe, in as much detail as possible, the structure of this image.

3.1 The diffeomorphism $\Omega^1(X_1 \sqcup X_2) = \Omega^1(X_1) \times \Omega^1(X_2)$

This is a rather easy and, in any case, expected fact, but for completeness we provide a proof.

Theorem 3.1. *The spaces $\Omega^1(X_1 \sqcup X_2)$ and $\Omega^1(X_1) \times \Omega^1(X_2)$ are diffeomorphic, for the usual functional diffeology on $\Omega^1(X_1 \sqcup X_2)$ and the product diffeology on $\Omega^1(X_1) \times \Omega^1(X_2)$.*

Proof. Let us first describe a bijection $\varphi : \Omega^1(X_1 \sqcup X_2) \rightarrow \Omega^1(X_1) \times \Omega^1(X_2)$. Let $\omega \in \Omega^1(X_1 \sqcup X_2)$, so that for every plot p of $X_1 \sqcup X_2$ there is a usual differential 1-form $\omega(p)$. Furthermore, every plot of X_1 is naturally a plot of $X_1 \sqcup X_2$ (and the same is true for every plot of X_2), therefore

$$\{\omega(p) \mid p \in \text{Plots}(X_1 \sqcup X_2)\} \supset \{\omega_1(p_1) \mid p_1 \in \text{Plots}(X_1)\},$$

where $\omega_1(p_1)$ is the differential 1-form (on the domain of definition of p_1) assigned by ω to the plot⁷ of $X_1 \sqcup X_2$ obtained by composing p_1 with the natural inclusion $X_1 \hookrightarrow X_1 \sqcup X_2$. Furthermore, there is an analogous inclusion for X_2 , that is,

$$\{\omega(p) \mid p \in \text{Plots}(X_1 \sqcup X_2)\} \supset \{\omega_2(p_2) \mid p_2 \in \text{Plots}(X_2)\}.$$

Notice, finally, that as sets,

$$\{\omega(p) \mid p \in \text{Plots}(X_1 \sqcup X_2)\} = \{\omega_1(p_1) \mid p_1 \in \text{Plots}(X_1)\} \cup \{\omega_2(p_2) \mid p_2 \in \text{Plots}(X_2)\};$$

indeed, it is a general property of the disjoint union diffeology (see [4], Ex. 22 on p.23) that for any plot $p : U \rightarrow X_1 \sqcup X_2$ we have $U = U_1 \cup U_2$, where $U_1 \cap U_2 = \emptyset$, and if U_i is non-empty then $p|_{U_i}$ is a plot of X_i . We indicate this fact by writing $\omega = \omega_1 \cup \omega_2$.

Observe now that each ω_i is a well-defined differential 1-form on X_i ; indeed, it is defined for all plots of X_i (these being plots of $X_1 \sqcup X_2$), and it satisfies the smooth compatibility condition simply because ω does. On the other hand, for any two forms ω_i on X_i their formal union $\omega_1 \cup \omega_2$ yields a differential form on $X_1 \sqcup X_2$, by the already-cited property of the disjoint union diffeology (since X_1 and X_2 are disjoint, the smooth compatibility condition is empty in this case). Thus, setting $\varphi(\omega_1 \cup \omega_2) = (\omega_1, \omega_2)$ yields a well-defined bijection $\Omega^1(X_1 \sqcup X_2) \leftrightarrow \Omega^1(X_1) \times \Omega^1(X_2)$; let us show that it is both ways smooth.

⁷We did not formally introduce the notation $\text{Plots}(X)$; its meaning as the set of all plots of X should be completely obvious.

Let $q : U' \rightarrow \Omega^1(X_1 \sqcup X_2)$ be a plot; we need to show that $\varphi \circ q$ is a plot of $\Omega^1(X_1) \times \Omega^1(X_2)$. Notice that each $q(u')$ writes in the form $q(u') = q_1(u') \cup q_2(u')$, and $(\varphi \circ q)(u') = (q_1(u'), q_2(u'))$. It suffices to show that each q_i , defined by $q_i(u')(p) = q(u)(p)$ whenever p coincides with a plot of X_i , is a plot of $\Omega^1(X_i)$. For it to be so, for any arbitrary plot $p_i : U_i \rightarrow X_i$ the evaluation map $U' \times U'_i \ni (u', u_i) \mapsto (q_i(u')(p_i))(u_i) \in \Lambda^1(U_i)$ should be smooth (in the usual sense). Now, the pair of plots p_1, p_2 defines a plot $p_1 \sqcup p_2 : U_1 \sqcup U_2 \rightarrow X_1 \sqcup X_2$ ⁸ of $X_1 \sqcup X_2$. The evaluation of $q(u')$ on this plot, smooth by hypothesis, is $(u', u_1) \mapsto (q(u')(p_1))(u_1) = (q_1(u')(p_1))(u_1)$ for $u_1 \in U_1$ and $(u', u_2) \mapsto (q(u')(p_2))(u_2) = (q_2(u')(p_2))(u_2)$ for $u_2 \in U_2$, by the definitions of q_1 and q_2 , so we are finished.

The proof works in a very similar manner for the inverse map φ^{-1} . Indeed, let $q_i : U' \rightarrow \Omega^1(X_i)$ for $i = 1, 2$ be a pair of plots of $\Omega^1(X_1)$, $\Omega^1(X_2)$ respectively; such a pair represents a plot of the direct product $\Omega^1(X_1) \times \Omega^1(X_2)$. We need to show that $\varphi^{-1} \circ (q_1, q_2) : U' \rightarrow \Omega^1(X_1 \sqcup X_2)$ is a plot of $\Omega^1(X_1 \sqcup X_2)$. Notice first of all that $(\varphi^{-1} \circ (q_1, q_2))(u') = q_1(u') \cup q_2(u')$. To show that the assignment $u' \mapsto (q_1(u') \cup q_2(u'))$ defines a plot of $\Omega^1(X_1 \sqcup X_2)$, consider a plot $p = p_1 \sqcup p_2 : U_1 \sqcup U_2 \rightarrow X_1 \sqcup X_2$ of $X_1 \sqcup X_2$ and the evaluation of $q_1(u') \cup q_2(u')$ on it. The same formulae as above show that we actually have a disjoint union of the evaluations of q_1 and q_2 , smooth by assumption, so we are finished. \square

3.2 The image of the pullback map $\Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1 \sqcup X_2)$

The pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1 \sqcup X_2)$ is defined by requiring, for any given a differential 1-form ω on $X_1 \cup_f X_2$, the form $\pi^*(\omega)$ to obey the following rule: if p is a plot of $X_1 \sqcup X_2$ then $(\pi^*(\omega))(p) = \omega(\pi \circ p)$ (see [4], Chapter 6). Following from this definition and from the already-established diffeomorphism $\Omega^1(X_1 \sqcup X_2) \cong \Omega^1(X_1) \times \Omega^1(X_2)$, each form in $\Omega^1(X_1 \cup_f X_2)$ splits as a pair of forms, one in $\Omega^1(X_1)$, the other in $\Omega^1(X_2)$. This allows us to show that the pullback map is *not* in general surjective, which we do in the section that follows, using the notion of an *f-invariant 1-form* and that of a pair of *compatible 1-forms*; these notions serve also to describe the image of the pullback map.

3.2.1 The map π^* composed with $\Omega^1(X_1 \sqcup X_2) \rightarrow \Omega^1(X_2)$ is surjective

The property, stated in the title of the section, follows easily from the existence of the induction $i_2^{X_2} : X_2 \rightarrow X_1 \cup_f X_2$ (see Section 1.3.2); the composition $[\Omega^1(X_1 \sqcup X_2) \rightarrow \Omega^1(X_2)] \circ \pi^*$ is the map $(i_2^{X_2})^*$, whose surjectivity follows from it being an induction.

3.2.2 Determining the projection to $\Omega^1(X_1)$: *f*-equivalent plots and *f*-invariant forms

As follows from the gluing construction, there is in general not an induction of X_1 into $X_1 \cup_f X_2$; the map $i_1^{X_1}$ is an induction, but it is defined on the proper subset $X_1 \setminus Y$ of X_1 . Obviously, there is a natural map $i'_1 : X_1 \rightarrow X_1 \cup_f X_2$ obtained by taking the composition of the inclusion $X_1 \hookrightarrow (X_1 \sqcup X_2)$ and the projection $(X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2$; in general, it is not an induction, so the corresponding pullback map $(i'_1)^*$ *a priori* is not surjective. It is rather clear that this is correlated to f being, or not, injective, so in general the forms in $\Omega^1(X_1)$ contained in the image of $(i'_1)^*$ should possess the property described in the second of the following definitions. We need an auxiliary term first.

Definition 3.2. Two plots p_1 and p'_1 of X_1 are said to be *f-equivalent* if they have the same domain of definition U and for all $u \in U$ such that $p_1(u) \neq p'_1(u)$ we have $p_1(u), p'_1(u) \in Y$ and $f(p_1(u)) = f(p'_1(u))$.

Thus, two plots on the same domain are *f*-equivalent if they differ only at points of the domain of gluing, and among such, only at those points that are identified by f .

Definition 3.3. A form $\omega_1 \in \Omega^1(X_1)$ is said to be *f-invariant* if for any two plots *f*-equivalent $p_1, p'_1 : U \rightarrow X_1$ we have $\omega_1(p_1) = \omega_1(p'_1)$.

As we will see with more precision below, this notion is designed to ensure that an *f*-invariant form descends to a well-defined form on the space resulting from gluing of X_1 to another diffeological space.

⁸The meaning of this notation is that $p(u) = p_1(u)$ for $u \in U_1$ and $p(u) = p_2(u)$ for $u \in U_2$ (we could also say that $p_i = p|_{U_i}$); the disjoint union $U_1 \sqcup U_2$ is considered as a disconnected domain in a Euclidean space large enough to contain both, and possibly applying a shift if both contain zero.

3.2.3 $\Omega_f^1(X_1)$ is a vector subspace of $\Omega^1(X_1)$

This is a consequence of the following statement.

Lemma 3.4. *Let $\omega'_1, \omega''_1 \in \Omega^1(X_1)$ be two f -invariant forms, and let $\alpha \in \mathbb{R}$. Then the forms $\omega'_1 + \omega''_1$ and $\alpha\omega'_1$ are f -invariant forms.*

Proof. Let $p', p'' : U \rightarrow X_1$ be two plots of X_1 with the following property: if $u \in U$ is such that $p'(u) \neq p''(u)$ then $p'(u), p''(u) \in Y$ and $f(p'(u)) = f(p''(u))$. The assumption that ω'_1, ω''_1 are f -invariant means that we have $\omega'_1(p') = \omega'_1(p'')$ and $\omega''_1(p') = \omega''_1(p'')$. The same equalities should now be checked for $\omega'_1 + \omega''_1$ and $\alpha\omega'_1$, and these follow immediately from the definition of the addition and scalar multiplication in $\Omega^1(X_1)$. Specifically,

$$\begin{aligned} (\omega'_1 + \omega''_1)(p') &= \omega'_1(p') + \omega''_1(p') = \omega'_1(p'') + \omega''_1(p'') = (\omega'_1 + \omega''_1)(p'') \text{ and} \\ (\alpha\omega'_1)(p') &= \alpha(\omega'_1(p')) = \alpha(\omega'_1(p'')) = (\alpha\omega'_1)(p''). \end{aligned}$$

□

We thus obtain that $\Omega_f^1(X_1)$ is a vector subspace of $\Omega^1(X_1)$. In particular, its intersection with any other vector subspace of $\Omega^1(X_1)$ is a vector subspace itself, so we always have a well-defined quotient (in the sense of vector spaces).

3.2.4 Characterizing the basic forms relative to π^*

The image of the map π^* can be easily described in the following terms.

Theorem 3.5. *Let ω_i be a differential 1-form on X_i , for $i = 1, 2$. The pair (ω_1, ω_2) belongs to the image of the pullback map π^* if and only if ω_1 is f -compatible, and for every plot p_1 of the subset diffeology on Y we have*

$$\omega_1(p_1) = \omega_2(f \circ p_1).$$

Proof. Suppose that $(\omega_1, \omega_2) = \pi^*(\omega)$ for some $\omega \in \Omega^1(X_1 \cup_f X_2)$. That ω_1 has to be f -compatible, has already been seen. Recall also that by definition $\omega_i(p_i) = \omega(\pi \circ p_i)$ for $i = 1, 2$ and any plot p_i of X_i .

Let us check that the second condition indicated in the statement holds. Let p_1 be a plot for the subset diffeology of Y ; then $f \circ p_1$ is a plot of X_2 . Furthermore, $\pi \circ p_1 = \pi \circ f \circ p_1$ by the very construction of $X_1 \cup_f X_2$. Therefore we have:

$$\omega_1(p_1) = \omega(\pi \circ p_1) = \omega(\pi \circ f \circ p_1) = \omega_2(f \circ p_1),$$

as wanted.

Let us now prove the reverse. Suppose that we are given two forms ω_1 and ω_2 , satisfying the condition indicated; let us define ω . Recall that, as we have already mentioned, it suffices to define ω on plots with connected domains. Let $p : U \rightarrow X_1 \cup_f X_2$ be such a plot; then it lifts either to a plot p_1 of X_1 or to a plot p_2 of X_2 . In the former case we define $\omega(p) = \omega_1(p_1)$, in the latter case we define $\omega(p) = \omega_2(p_2)$. Finally, if p is defined on a disconnected domain, $\omega(p)$ is defined by the collection of the values of its restriction to the corresponding connected components.

Let us show that this definition is well-posed (which it may not be *a priori* if p happens to lift to two distinct plots). Now, if p lifts to a plot of X_2 then this lift is necessarily unique, since $i_2 : X_2 \rightarrow X_1 \cup_f X_2$ is an induction. Suppose now that p lifts to two distinct plots $p_1 : U \rightarrow X_1$ and $p'_1 : U \rightarrow X_1$ of X_1 . It is then clear that p_1 and p'_1 differ only at points of Y , and among such, only at those that have the same image under f . More precisely, for any $u \in U$ such that $p_1(u) \neq p'_1(u)$, we have $p_1(u), p'_1(u) \in Y$ and $f(p_1(u)) = f(p'_1(u))$. Thus, for ω to be well-defined we must have that $\omega_1(p_1)(u) = \omega_1(p'_1)(u)$ for all such u .

What we now need to check is whether ω thus defined satisfies the smooth compatibility condition. Let $q : U' \rightarrow X_1 \cup_f X_2$ be another plot of $X_1 \cup_f X_2$ for which there exists a smooth map $g : U' \rightarrow U$ such that $q = p \circ g$. We need to check that $\omega(q) = g^*(\omega(p))$.

Suppose first that p lifts to a plot p_2 of X_2 . Then we have $p = \pi \circ p_2$, so $q = \pi \circ p_2 \circ g$. Notice that $p_2 \circ g$ is also a plot of X_2 and is a lift of q ; thus, according to our definition $\omega(q) = \omega_2(p_2 \circ g) = g^*(\omega_2(p_2)) = g^*(\omega(p))$. If now p lifts to a plot p_1 of X_1 the same argument is sufficient, whence the conclusion. □

The theorem just proved motivates the following definition, which will serve to characterize the basic forms in $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$.

Definition 3.6. Let $\omega_i \in \Omega^1(X_i)$ for $i = 1, 2$. We say that ω_1 and ω_2 are **compatible** with respect to the gluing along f if for every plot p_1 of the subset diffeology on the domain Y of f we have

$$\omega_1(p_1) = \omega_2(f \circ p_1).$$

3.3 The pullback map is a diffeomorphism $\Omega^1(X_1 \cup_f X_2) \cong \Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$

We now obtain our first definite conclusion regarding the space $\Omega^1(X_1 \cup_f X_2)$; namely, in this section we construct a smooth inverse of the map π^* , which obviously ensures the claim in the title of the section.

3.3.1 Constructing the inverse of π^*

Let us first define this map; in the next section we will prove that it is smooth.

The induced 1-form $\omega_1 \cup_f \omega_2$ on $X_1 \cup_f X_2$ Let ω_1 be an f -invariant 1-form on X_1 , and let ω_2 be a 1-form on X_2 such that ω_1 and ω_2 are compatible. Let $p : U \rightarrow X_1 \cup_f X_2$ be an arbitrary plot of $X_1 \cup_f X_2$; the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is defined as follows.

Let $u \in U$; in any connected neighborhood of $x = p(u)$ the plot p lifts to either a plot p_1 of X_1 or a plot p_2 of X_2 . We define, accordingly,

$$(\omega_1 \cup_f \omega_2)(p)(u) := \omega_i(p_i)(u).$$

Lemma 3.7. If ω_1 is f -invariant, and ω_1 and ω_2 are compatible with each other, the differential 1-form $\omega_1 \cup_f \omega_2$ is well-defined.

Proof. We need to show that for each plot $p : U \rightarrow X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is well-defined, i.e., that it does not depend on the choice of the lift of the plot p . Obviously, it suffices to assume that U is connected, which then implies that p lifts either to a plot of X_1 or to a plot of X_2 . If p has a unique lift, then there is nothing to prove. Suppose that p has two distinct lifts, p' and p'' . Notice that X_2 smoothly injects into $X_1 \cup_f X_2$, therefore p' and p'' cannot be both plots of X_2 .

Assume first that one of them, say p' , is a plot of X_1 , while the other, p'' , is a plot of X_2 . Since they project to the same map to $X_1 \cup_f X_2$, we can conclude that $p'' = f \circ p'$, so

$$(\omega_1 \cup_f \omega_2)(p) = \omega_1(p') = \omega_2(f \circ p') = \omega_2(p''),$$

by the compatibility of the forms ω_1 and ω_2 with each other.

Assume now that p' and p'' are both plots of X_1 . Once again, since they project to the same plot of $X_1 \cup_f X_2$, for every $u \in U$ such that $p'(u) \neq p''(u)$ we have $p'(u), p''(u) \in Y$ and $f(p'(u)) = f(p''(u))$, that is, that they are f -equivalent; since ω_1 is assumed to be f -invariant, we obtain that

$$\omega_1(p') = \omega_1(p'') = (\omega_1 \cup_f \omega_2)(p).$$

We can thus conclude that for each plot $p : U \rightarrow X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is well-defined. It remains to observe that the resulting collection $\{(\omega_1 \cup_f \omega_2)(p)\}$ of usual differential 1-forms satisfies the smooth compatibility condition for diffeological differential forms for all the same reasons as those given at the end of the proof of Theorem 3.5. \square

The map $\mathcal{L} : \Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega^1(X_1 \cup_f X_2)$ As we have seen above, the assignment

$$(\omega_1, \omega_2) \mapsto \omega_1 \cup_f \omega_2$$

to any two compatible differential 1-forms $\omega_1 \in \Omega_f^1(X_1)$ and $\omega_2 \in \Omega^1(X_2)$, of the differential 1-form $\omega_1 \cup_f \omega_2$ is well-defined. This yields a map \mathcal{L} defined on the set $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$ of all pairs of compatible 1-forms (such that the first component of the pair is f -invariant), with range the space of 1-forms on $X_1 \cup_f X_2$.

Lemma 3.8. *The map \mathcal{L} is the inverse of the pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1) \times \Omega^1(X_2)$.*

Proof. This follows from construction. Indeed, let $\omega \in \Omega^1(X_1 \cup_f X_2)$; recall that $\pi^*(\omega) = (\omega_1, \omega_2)$, where for every plot p_i of X_i we have $\omega_i(p_i) = \omega(\pi \circ p_i)$. Furthermore, by Theorem 3.5, the form ω_1 is f -invariant and the two forms ω_1 and ω_2 are compatible with each other. Therefore the pair (ω_1, ω_2) is in the domain of \mathcal{L} , and by construction $\omega_1 \cup_f \omega_2$ is precisely ω .

Vice versa, let (ω_1, ω_2) be in the domain of definition of \mathcal{L} . It suffices to observe that $\pi^*(\omega_1 \cup_f \omega_2) = (\omega'_1, \omega'_2)$ with $\omega'_i(p_i) = (\omega_1 \cup_f \omega_2)(\pi \circ p_i) = \omega_i(p_i)$ for any plot p_i of X_i , which means that $\omega'_i = \omega_i$ for $i = 1, 2$. \square

3.3.2 The inverse of the pullback map is smooth

To prove that the map π^* is a diffeomorphism, it remains to show that its inverse \mathcal{L} is a smooth map, for the standard diffeologies on its domain and its range. Specifically, the range carries the standard diffeology of the space of 1-forms on a diffeological space (see Section 2), while the domain is endowed with the subset diffeology relative to the inclusion $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \subset \Omega^1(X_1) \times \Omega^1(X_2)$ (this direct product has, as usual, the product diffeology relative to the standard diffeologies on $\Omega^1(X_1)$ and $\Omega^1(X_2)$).

Theorem 3.9. *The map $\mathcal{L} : \Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega^1(X_1 \cup_f X_2)$ is smooth.*

Proof. Consider a plot $p : U \rightarrow \Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$. First of all, by definition of a subset diffeology and a product diffeology, we can assume that U is small enough so that for every $u \in U$ we have $p(u) = (p_1(u), p_2(u))$, where $p_1 : U \rightarrow \Omega_f^1(X_1)$ is a plot of $\Omega_f^1(X_1)$ (considered with the subset diffeology relative to the inclusion $\Omega_f^1(X_1)$), $p_2 : U \rightarrow \Omega^1(X_2)$ is a plot of $\Omega^1(X_2)$, and $p_1(u)$ and $p_2(u)$ are compatible with respect to f , for all $u \in U$.

That p_i is a plot of $\Omega^1(X_i)$, by definition of the standard diffeology on the latter, means that for every plot $q_i : U'_i \rightarrow X_i$ the map $U \times U'_i \rightarrow \Lambda^1(\mathbb{R})$, acting by $(u, u'_i) \mapsto (p_i(u))(q_i)(u'_i)$, is smooth (in the usual sense). The compatibility of the forms $p_1(u)$ and $p_2(u)$ means that $p_2(u)(f \circ q_1) = p_1(u)(q_1)$, for all u and for all plots q_1 of the subset diffeology of Y .

Suppose we are given p_1 and p_2 satisfying all of the above. We need to show that $(\mathcal{L} \circ (p_1, p_2)) : U \rightarrow \Omega^1(X_1 \cup_f X_2)$ is a plot of $\Omega^1(X_1 \cup_f X_2)$. This, again, amounts to showing that for any plot $q : U' \rightarrow X_1 \cup_f X_2$ the evaluation map

$$(u, u') \mapsto (\mathcal{L}(p_1(u), p_2(u)))(q)(u')$$

defined a usual smooth map $U \times U' \rightarrow \Lambda^1(\mathbb{R}^n)$ (for $U' \subset \mathbb{R}^n$).

Assume that U' is connected so that q lifts either to a plot q_1 of X_1 , or a plot q_2 of X_2 . It may furthermore lift to more than one plot of X_1 , or it may lift to both a plot of X_1 and a plot of X_2 . Suppose first that q lifts to a precisely one plot, say a plot q_i of X_i . Then

$$(u, u') \mapsto (\mathcal{L}(p_1(u), p_2(u)))(q)(u') = p_i(u)(q_i)(u') \in \Lambda^1(\mathbb{R}^n);$$

this is a smooth map, since each p_i is a plot of $\Omega^1(X_i)$.

Suppose now that q lifts to two distinct plots q_1 and q'_1 of X_1 . In this case, however, $p_1(u)(q_1) = p_1(u)(q'_1)$ because $p_1(u)$ is f -compatible for any $u \in U$ by assumption, so we get the desired conclusion as in the previous case. Finally, if q lifts to both q_1 and q_2 (each q_i being a plot of X_i) then $q_2 = f \circ q_1$, and we obtain the claim by using the compatibility of the pair of forms $p_1(u), p_2(u)$ for each u . \square

Corollary 3.10. *The pullback map π^* is a diffeomorphism with its image.*

Proof. We have just seen that π^* has a smooth inverse \mathcal{L} . It remains to observe that π^* itself is smooth, because the pullback of a smooth map is always smooth itself (see [4], Section 6.38). \square

4 Reduction to the case of gluing along a diffeomorphism: substituting $\Omega_f^1(X_1)$ with $\Omega^1(X_1^f)$

What we mean by a sub-direct product⁹ of any direct product $X \times Y$ of two sets is any subset such that both projections on the two factors X and Y are surjective. Thus, the question of whether $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$ is a sub-direct product of $\Omega_f^1(X_1)$ and $\Omega^1(X_2)$ takes the form of the following two: first, if $\omega_1 \in \Omega_f^1(X_2)$ is an arbitrary f -invariant form, does there always exist $\omega_2 \in \Omega^1(X_2)$ compatible with it?, and *vice versa*, if $\omega_2 \in \Omega^1(X_2)$ is any form, does there exist an f -invariant form ω_1 on X_1 , compatible with ω_2 as well? An additional assumption on f , namely, that it is a subduction, ensures that the answer is positive in both cases. Proving this requires an intermediate construction.

4.1 The space of f -equivalence classes X_1^f

The intermediate construction just mentioned is a certain auxiliary space X_1^f , which is a quotient of X_1 . The aim of introducing it is to identify the space $X_1 \cup_f X_2$ with a result of a specific gluing of X_1^f to X_2 ; and under the assumption that f is a subduction, this other gluing turns out to be a diffeomorphism.

The space X_1^f and the map f_\sim In order to consider the projection $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$, we introduce a slightly different form of our glued space $X_1 \cup_f X_2$. Let us define X_1^f to be the diffeological¹⁰ quotient of X_1 by the equivalence relation $y_1 \sim y_2 \Leftrightarrow f(y_1) = f(y_2)$, that is:

$$X_1^f := X_1 / (f(y_1) = f(y_2)).$$

Let $\pi_1^f : X_1 \rightarrow X_1^f$ be the quotient projection, and let us define the map $f_\sim : X_1^f \supseteq \pi_1^f(Y) \rightarrow X_2$ induced by f . This map is given by the condition $f_\sim \circ \pi_1^f = f$.

Lemma 4.1. *The map f_\sim is injective and smooth. It is a diffeomorphism with its image if and only if f is a subduction.*

Proof. The injectivity of f_\sim is by construction (we actually defined the space X_1^f so that the pushforward of f to it be injective), and its smoothness follows from the definition of the quotient diffeology. Recall now that a subduction is a smooth map such that the diffeology on its target space is the pushforward of the diffeology of its domain by the map. Thus, the assumption that f , considered as a map $Y \rightarrow f(Y)$, is a subduction means that for every plot q of the subset diffeology on $f(Y)$, defined on a sufficiently small neighborhood, there is a plot p of the subset diffeology on Y such that $f \circ p = q$. Therefore $f_\sim \circ (\pi_1^f \circ p) = q$, and so $(f_\sim)^{-1} \circ q = \pi_1^f \circ p$ for any plot q of $f(Y)$ and for an appropriate plot p of Y . Since $\pi_1^f \circ p$ is a plot of $\pi_1^f(Y)$, we conclude that $(f_\sim)^{-1}$ is smooth, and so f_\sim is a diffeomorphism with its image. We obtain the *vice versa* by applying the same reasoning in the reverse order. \square

Lifts of plots of X_1^f By definition of the quotient diffeology, every plot p of X_1^f lifts (locally) to a plot of X_1 . Two lifts p' and p'' are lifts of the same p if and only if they are f -equivalent.

The diffeomorphism $X_1 \cup_f X_2 \cong X_1^f \cup_{f_\sim} X_2$ The existence of this diffeomorphism is a direct consequence of the definition of gluing. Formally, it is defined as the pushforward of the map $\pi_1^f \sqcup \text{Id}_{X_2} : X_1 \sqcup X_2 \rightarrow X_1 \sqcup X_2$ by the two quotient projections, π and π^f respectively. Here by $\pi_1^f \sqcup \text{Id}_{X_2}$ we mean the map on $X_1 \sqcup X_2$, whose value at an arbitrary point $x \in X_1 \sqcup X_2$ is $\pi_1^f(x)$ if $x \in X_1$ and x if $x \in X_2$; the map $\pi^f : X_1^f \sqcup X_2 \rightarrow X_1^f \cup_{f_\sim} X_2$ is, as we said, the quotient projection that defines the space $X_1^f \cup_{f_\sim} X_2$.

⁹Which is probably more or less a standard notion.

¹⁰That is, endowed with the quotient diffeology.

4.2 The linear diffeomorphism $\Omega_f^1(X_1) \cong \Omega^1(X_1^f)$

The reason that explains the introduction of the space X_1^f is that it allows to consider, instead of a subset of 1-forms on X_1 , the space of all 1-forms on X_1^f ; and to obtain $X_1 \cup_f X_2$ by gluing X_1^f to X_2 along a bijective map (a diffeomorphism if we assume f to be a subduction, see above).

Proposition 4.2. *The pullback map $(\pi_1^f)^* : \Omega^1(X_1^f) \rightarrow \Omega_f^1(X_1)$ is a diffeomorphism.*

Proof. Let us first show that $(\pi_1^f)^*$ takes values in $\Omega_f^1(X_1)$. Let $\omega \in \Omega^1(X_1^f)$; its image $(\pi_1^f)^*(\omega)$ is defined by setting, for every plot p_1 of X_1 , that $(\pi_1^f)^*(\omega)(p_1) = \omega(\pi_1^f \circ p_1)$. We need to show that $(\pi_1^f)^*(\omega)$ is f -invariant, so let p_1 and p'_1 be two f -equivalent plots; then we have $\pi_1^f \circ p_1 = \pi_1^f \circ p'_1$, and so $(\pi_1^f)^*(\omega)(p_1) = (\pi_1^f)^*(\omega)(p'_1)$. Thus, the range of $(\pi_1^f)^*$ is contained in $\Omega_f^1(X_1)$.

Let us show $(\pi_1^f)^*$ is a bijection by constructing its inverse. Let ω_1 be an f -invariant 1-form on X_1 , and let us assign to it a form ω_1^f on X_1^f by setting $\omega_1^f(p_1^f) = \omega_1(p_1)$, where p_1 is any lift to X_1 of the plot p_1^f . We need to show that this is well-defined, *i.e.*, $\omega_1(p_1)$ does not depend on the choice of a specific lift.¹¹ Indeed, let p_1 and p'_1 be two lifts of some p_1^f ; this means, first, that they have the same domain of definition U and, second, that for any $u \in U$ such that $p_1(u) \neq p'_1(u)$, we have $p_1(u), p'_1(u) \in Y$ and $f(p_1(u)) = f(p'_1(u))$. In other words, they are f -equivalent, so by f -invariance of ω_1 we have $\omega_1(p_1) = \omega_1(p'_1)$. The form ω_1^f is therefore well-defined, and the fact that the assignment $\Omega_f^1 \ni \omega_1 \mapsto \omega_1^f \in \Omega^1(X_1^f)$ is obvious from the construction.

Thus, $(\pi_1^f)^*$ is a bijective map and, as any pullback map, it is smooth. It thus remains to show that its inverse, that we have just constructed, is smooth (with respect to the usual functional diffeology of a space of forms; obviously, the diffeology of $\Omega_f^1(X_1)$ is the subset diffeology relative to its inclusion in $\Omega^1(X_1)$).

Let $q : U' \rightarrow \Omega_f^1(X_1)$ be a plot of $\Omega_f^1(X_1)$; thus, for every plot $p_1 : \mathbb{R}^n \supset U \rightarrow X_1$ the evaluation map $(u', u) \mapsto (q(u')(p))(u)$ is a smooth map to $\Lambda^1(\mathbb{R}^n)$, and furthermore for any $u' \in U'$ and for any two f -equivalent plots p_1, p'_1 of X_1 , *i.e.*, such that $\pi_1^f \circ p_1 = \pi_1 \circ p'_1$, we have $q(u')(p_1) = q(u')(p'_1)$. Let us now consider the composition $\left((\pi_1^f)^*\right)^{-1} \circ q$; as always, we need to show that this is a plot of $\Omega^1(X_1^f)$. Since the plots of X_1^f are defined by classes of f -equivalent plots of X_1 , and the forms $\left(\left((\pi_1^f)^*\right)^{-1} \circ q\right)(u')$ are given by values of $q(u')$ on class representatives, the evaluation map for this composition is simply the same as the one for q , so we get the desired conclusion. \square

Thus, the f -invariant forms on X_1 are precisely the pullbacks by the natural projection of the forms on X_1^f . Furthermore, by construction of X_1^f we can, instead of gluing between X_1 and X_2 , consider the corresponding gluing between X_1^f and X_2 , which has an advantage of being a gluing along a bijective map.

4.3 The space $\Omega^1(X_1^f \cup_{f\sim} X_2)$

We have already given a description of the space $\Omega^1(X_1 \cup_f X_2)$ in terms of $\Omega_f^1(X_1)$ and $\Omega^1(X_2)$. We now use the presentation of $X_1 \cup_f X_2$ as $X_1^f \cup_{f\sim} X_2$, to write the same space $\Omega^1(X_1 \cup_f X_2) = \Omega^1(X_1^f \cup_{f\sim} X_2)$ in terms of $\Omega^1(X_1^f)$ and $\Omega^1(X_2)$.

Compatibility of $\omega_1 \in \Omega^1(X_1^f)$ and $\omega_2 \in \Omega^1(X_2)$ The notion of compatibility admits an obvious extension to the case of a 1-form $\omega_1^f \in \Omega^1(X_1^f)$ and a 1-form $\omega_2 \in \Omega^1(X_2)$. This notion is the same as the f -compatibility, but considered with respect to $f\sim$. Specifically, ω_1^f and ω_2 are said to be $f\sim$ -**compatible** if for every plot p_1^f of $Y^f = \pi_1^f(Y)$, considered with the subset diffeology relative to the inclusion $Y^f \subseteq X_1^f$ we have

$$\omega_1^f(p_1^f) = \omega_2(f\sim \circ p_1^f).$$

We then easily obtain the following.

¹¹At least one lift always exists, by the properties of the quotient diffeology.

Lemma 4.3. *The forms ω_1^f and ω_2 are f_\sim -compatible if and only if $\omega_1 = (\pi_1^f)^*(\omega_1^f)$ and ω_2 are f -compatible.*

Proof. Let $p_1^f : U \rightarrow X_1^f$ be a plot of X_1^f , and let $\{p_1^{(i)}\}$ be the collection of all its lifts to X_1 ; this collection is then an equivalence class by f -equivalence, and moreover, we always have

$$f_\sim \circ p_1^f = f \circ p_1^{(i)}.$$

This ensures that the equalities $\omega_1^f(p_1^f) = \omega_2(f \circ p_1^{(i)})$ and $\omega_1(p_1^{(i)}) = \omega_2(f \circ p_1^{(i)})$ hold simultaneously, whence the claim. \square

The diffeomorphism $\Omega^1(X_1 \cup_f X_2) \cong \Omega^1(X_1^f) \times_{\text{comp}} \Omega^1(X_2)$ The lemma just proven, together with Proposition 4.2, trivially imply that

$$\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \cong \Omega^1(X_1^f) \times_{\text{comp}} \Omega^1(X_2).$$

This, together with Corollary 3.10 (and Theorem 3.9), yields immediately the following.

Proposition 4.4. *There is a natural diffeomorphism*

$$\Omega^1(X_1 \cup_f X_2) \cong \Omega^1(X_1^f) \times_{\text{comp}} \Omega^1(X_2),$$

that filters through the pullback map π^ .*

Notice that this diffeomorphism is given by

$$((\pi_1^f)^* \times \text{Id}_{X_2}^*) \circ [\Omega^1(X_1 \sqcup X_2) \rightarrow \Omega^1(X_1) \times \Omega^1(X_2)] \circ \pi^*.$$

5 The images of the projections $\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega^1(X_1)$ and $\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$

From this section onwards, we assume that f is a diffeomorphism of its domain with its image. Furthermore, in the section that follows and behind, we will add another assumption, namely the equality

$$i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2)),$$

whose meaning, stemming from the compatibility notion, we now explain.

5.1 Re-interpreting the compatibility

We now give an equivalent formulation of the compatibility notion for $\omega_1 \in \Omega^1(X_1)$ and $\omega_2 \in \Omega^1(X_2)$, using three types of the pullback map. Specifically, let $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$ be the natural inclusion maps, and let $i^* : \Omega^1(X_1) \rightarrow \Omega^1(Y)$ and $j^* : \Omega^1(X_2) \rightarrow \Omega^1(f(Y))$ be the corresponding pullback maps. Then there is the following statement (that is true without any extra assumptions on f).

Proposition 5.1. *Let $\omega_i \in \Omega^1(X_i)$ for $i = 1, 2$. Then ω_1 and ω_2 are compatible if and only if we have*

$$f^*(j^*\omega_2) = i^*\omega_1.$$

Proof. Suppose first that ω_1 and ω_2 are compatible; consider $f^*(j^*\omega_2)$ and $i^*\omega_1$, both of which belong to $\Omega^1(Y)$. Let $p : U \rightarrow Y$ be a plot for the subset diffeology of Y ; then

$$i^*(\omega_1)(p) = \omega_1(i \circ p) = \omega_1(p),$$

where we identify the plot p with $i \circ p$, as is typical for the plots in a subset diffeology. Likewise,

$$f^*(j^*\omega_2)(p) = j^*(\omega_2)(f \circ p) = \omega_2(j \circ (f \circ p)) = \omega_2(f \circ p),$$

where again we identify $j \circ (f \circ p)$ and $f \circ p$. By the compatibility of ω_1 and ω_2 , we have that $\omega_1(p) = \omega_2(f \circ p)$, which implies that $f^*(j^*(\omega_2))(p) = i^*(\omega_1)(p)$ for all plots in the subset diffeology of Y ; this means precisely that $f^*(j^*(\omega_2))$ and $i^*(\omega_1)$ are equal as forms in $\Omega^1(Y)$.

The *vice versa* of this statement is obtained from the same two equalities, by assuming first that $f^*(j^*(\omega_2))(p) = i^*(\omega_1)(p)$ for all p and concluding that then also $\omega_1(p) = \omega_2(f \circ p)$, which is the condition for the compatibility of forms ω_1 and ω_2 . \square

The proposition just proven allows us to give an alternative description of the subspace $\Omega^1(X_1) \times_{comp} \Omega^1(X_2) \cong \pi^*(X_1 \cup_f X_2)$, which is as follows.

Corollary 5.2. *Let $(i^*, f^*j^*) : \Omega^1(X_1) \times \Omega^1(X_2) \rightarrow \Omega^1(Y) \times \Omega^1(Y)$ be the direct product map. Then*

$$\pi^*(\Omega^1(X_1 \cup_f X_2)) \cong \Omega^1(X_1) \times_{comp} \Omega^1(X_2) = (i^*, f^*j^*)^{-1}(\text{diag}(\Omega^1(Y) \times \Omega^1(Y))).$$

The space $\text{diag}(\Omega^1(Y) \times \Omega^1(Y))$ is the usual diagonal of the direct product $\Omega^1(Y) \times \Omega^1(Y)$.

5.2 The images of the two projections

The criterion of the compatibility of forms stated in Proposition 5.1 allows us, in turn, to state the condition for the surjectivity of the projections of $\Omega^1(X_1) \times_{comp} \Omega^1(X_2)$ to its factors. We denote these projections by pr_1 (in the case of the first factor) and pr_2 (in the case of the second factor).

Proposition 5.3. *The two projections $pr_1 : \Omega^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega^1(X_1)$ and $pr_2 : \Omega^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$ are both surjective if and only if the following is true:*

$$i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2)).$$

Proof. Consider pr_1 . Its image is the set of all forms $\omega_1 \in \Omega^1(X_1)$ such that there exists at least one form $\omega_2 \in \Omega^1(X_2)$ compatible with ω_1 . Since by Proposition 5.1 this is equivalent to $i^*\omega_1 = f^*(j^*\omega_2)$, the existence of ω_2 is equivalent to $i^*\omega_1 \in (f^*j^*)(\Omega^1(X_2))$. \square

Remark 5.4. *In general, the image of the projection pr_1 is the subspace $(i^*)^{-1}((f^*j^*)(\Omega^1(X_2)))$, and vice versa, the image of pr_2 is the subspace $(j^*)^{-1}(f^*)^{-1}(i^*(\Omega^1(X_1)))$. Given the breadth of the notion of a diffeological space, we prefer not to look for alternative characterizations of their surjectivity.*

6 Vanishing 1-forms and the pullback map π^*

Recall that the pseudo-bundle $\Lambda^1(X)$, for any diffeological space X , is defined via quotienting over the collection of subspaces of forms vanishing at the given point. Thus, in this section we consider how the vanishing of forms interacts with the pullback map π^* .

Let $x \in X_1 \cup_f X_2$, and let $\omega \in \Omega^1(X_1 \cup_f X_2)$. By definition, ω vanishes at x if for every plot $p : U \rightarrow X_1 \cup_f X_2$ such that $U \ni 0$ and $p(0) = x$ we have $\omega(p)(0) = 0$. Let us consider the pullback form $\pi^*(\omega)$, written as $\pi^*(\omega) = (\omega_1, \omega_2)$; the fact that ω vanishes at some point x might then imply that either ω_1 or ω_2 , or both, vanish at one of, or all, lifts of x ; and going still furthermore, some kind of a reverse of this statement might hold. Below we discuss precisely this kind of question, concentrating on the structure of the pullback of a form on $X_1 \cup_f X_2$ vanishing at some x . Three cases arise there, that depend on the nature of x .

6.1 The pullbacks of forms on $X_1 \cup_f X_2$ vanishing at a point

Let $x \in X_1 \cup_f X_2$, and let $\omega \in \Omega^1(X_1 \cup_f X_2)$ be a form vanishing at x . It is quite obvious then (but worth stating anyhow) that the pullback of ω vanishes at any lift of this point.

Lemma 6.1. *Let $x \in X_1 \cup_f X_2$, let $\omega \in \Omega^1(X_1 \cup_f X_2)$ be a form vanishing at x , and let $\pi^*(\omega) = (\omega_1, \omega_2)$. Let $\tilde{x} \in X_i$ be such that $\pi(\tilde{x}) = x$. Then the corresponding ω_i vanishes at \tilde{x} .*

Proof. Let $p_i : U \rightarrow X_i$ be a plot centered at \tilde{x} ; then obviously, $\pi \circ p_i$ is a plot of $X_1 \cup_f X_2$ centered at x . Furthermore, $\omega_i(p)(0) = \omega(\pi \circ p)(0) = 0$, since ω vanishes at x . \square

Let us consider the implications of this lemma. Note first of all that at this point it is convenient to consider X_1^f instead of X_1 , identifying $\Omega_f^1(X_1)$ with $\Omega^1(X_1)$ and, whenever it is convenient, the space $X_1 \cup_f X_2$ with $X_1^f \cup_{f\sim} X_2$. Recall that π^f stands for the obvious projection $X_1 \sqcup X_2 \rightarrow X_1^f \cup_{f\sim} X_2$, and let $x \in X_1^f \cup_{f\sim} X_2$; there are three cases (two of which are quite similar).

If $x \in i_1(X_1 \setminus Y)$ then it has a unique lift, both with respect to π^f and with respect to π ; this lift furthermore is contained in X_1^f and X_1 respectively. Therefore

$$\begin{aligned} \pi^*(\Omega_x^1(X_1 \cup_f X_2)) &\cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f\sim} X_2)) \subseteq \\ &\subseteq \Omega_{(\pi^f)^{-1}(x)}^1(X_1^f) \times_{\text{comp}} \Omega^1(X_2) \cong (\Omega_f^1)_{\pi^{-1}(x)}(X_1) \times_{\text{comp}} \Omega^1(X_2). \end{aligned}$$

The case when $x \in i_2(X_2 \setminus f(Y))$ is similar; the lift of x is also unique then, and belongs to $X_2 \setminus f(Y)$. We thus have a similar sequence of inclusions:

$$\begin{aligned} \pi^*(\Omega_x^1(X_1 \cup_f X_2)) &\cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f\sim} X_2)) \subseteq \\ &\subseteq \Omega^1(X_1^f) \times_{\text{comp}} \Omega_{(\pi^f)^{-1}(x)}^1(X_2) \cong \Omega_f^1(X_1) \times_{\text{comp}} \Omega_{\pi^{-1}(x)}^1(X_2). \end{aligned}$$

Now, in the third case, which is the one of $x \in \pi^f(Y)$, it admits precisely two lifts via π^f , one to a point $\tilde{x}_1 \in X_1^f$, the other to a point $\tilde{x}_2 \in X_2$. By Lemma 6.1,

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f\sim} X_2)) \subseteq \Omega_{\tilde{x}_1}^1(X_1^f) \times_{\text{comp}} \Omega_{\tilde{x}_2}^1(X_2);$$

note that in this case, \tilde{x}_1 , which is a point of X_1^f , may have multiple (possibly infinitely many) lifts to X_1 .

6.2 Classification of pullback spaces according to the point of vanishing

The discussion carried out in the section immediately above leads the following statement.

Proposition 6.2. *Let $x \in X_1 \cup_f X_2$. Then:*

1. *If $x \in i_1(X_1 \setminus Y)$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \subseteq (\Omega_f^1)_{\pi^{-1}(x)}(X_1) \times_{\text{comp}} \Omega^1(X_2).$$

2. *If $x \in i_2(X_2 \setminus f(Y))$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \subseteq \Omega_f^1(X_1) \times_{\text{comp}} \Omega_{\pi^{-1}(x)}^1(X_2).$$

3. *If $x \in \pi(Y) = i_2(f(Y))$, and $\tilde{x}_1 \in X_1^f$ and $\tilde{x}_2 \in X_2$ are the two points in $(\pi^f)^{-1}(x)$, then*

$$(\pi^f)^*(\Omega_x^1(X_1 \cup_f X_2)) \subseteq \Omega_{\tilde{x}_1}^1(X_1^f) \times_{\text{comp}} \Omega_{\tilde{x}_2}^1(X_2).$$

The questions that arise now are, whether any, or all, of the three inclusions are actually identities, and, for the third item, how the space $\Omega_{\tilde{x}_1}^1(X_1^f)$ is related to one or more spaces of vanishing f -compatible forms on X_1 .

6.3 The reverse inclusion for points in $i_2(X_2 \setminus f(Y))$ and $i_1(X_1 \setminus Y)$

This follows from a rather simple observation. If $\omega_2 \in \Omega^1(X_2)$ is a form that also belongs to the image of the projection $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$ (that is, there exists an f -compatible form ω_1 on X_1 such that ω_1 and ω_2 are compatible between them), and ω_2 vanishes at some point x_2 , then any form on $X_1 \cup_f X_2$ to which ω_2 projects, also vanishes, at the point of $X_1 \cup_f X_2$ that corresponds to x_2 .

Lemma 6.3. *Let $(\omega_1, \omega_2) \in \Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$, and let $\tilde{x} \in X_2$ be such that ω_2 vanishes at \tilde{x} . Then $\omega_1 \cup_f \omega_2$ vanishes at $x := \pi(\tilde{x})$.*

Proof. Let $p : U \rightarrow X_1 \cup_f X_2$ be a plot centered at x . As we have noted above, \tilde{x} is the only lift of x to X_2 (although it may have lifts to X_1 as well), and any lift of p to a plot of X_2 is centered at \tilde{x} . Note also that at least one such lift exists, by definition of a pushforward diffeology and the disjoint union diffeology on $X_1 \sqcup X_2$. It remains to observe that if p_2 is such a lift then by construction $(\omega_1 \cup_f \omega_2)(p)(0) = \omega_2(p_2)(0) = 0$, whence the claim. \square

This lemma, together with Proposition 5.3 and the second point of Proposition 6.2, immediately implies the following.

Corollary 6.4. *If $(f^*j^*)(\Omega^1(X_2)) \subseteq i^*(\Omega^1(X_1))$ and $x \in i_2(X_2 \setminus f(Y))$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) = \Omega_f^1(X_1) \times_{\text{comp}} \Omega_{\pi^{-1}(x)}^1(X_2).$$

The case of a point in $i_1(X_1 \setminus Y)$ is completely analogous to that of a point in $i_2(X_2 \setminus f(Y))$, since the main argument is based on the same property, that of there being a unique lift of the point of vanishing. We therefore immediately state the final conclusion.

Corollary 6.5. *If $i^*(\Omega^1(X_1)) \subseteq (f^*j^*)(\Omega^1(X_2))$ and $x \in i_1(X_1 \setminus Y)$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) = (\Omega_f^1)_{\pi^{-1}(x)}(X_1) \times_{\text{comp}} \Omega^1(X_2).$$

6.4 The case of points in $\pi(Y) = i_2(f(Y))$

For points such as these, we already have the inclusion $(\pi^f)^*(\Omega_x^1(X_1 \cup_f X_2)) \subseteq \Omega_{\tilde{x}_1}^1(X_1^f) \times_{\text{comp}} \Omega_{\tilde{x}_2}^1(X_2)$. The questions to consider now are, first, whether it is actually an identity, and then, how the right-hand side is related to one or more subspaces of f -invariant forms on X_1 vanishing at points in the lift of x .

6.4.1 The reverse inclusion $(\pi^f)^*(\Omega_x^1(X_1 \cup_f X_2)) \supseteq \Omega_{\tilde{x}_1}^1(X_1^f) \times_{\text{comp}} \Omega_{\tilde{x}_2}^1(X_2)$

Let $x \in X_1 \cup_f X_2$, and let $\tilde{x}_1 \in X_1^f$, $\tilde{x}_2 \in X_2$ be such that $\pi^f(\tilde{x}_i) = x$, which is equivalent to $f_{\sim}(\tilde{x}_1) = \tilde{x}_2$. Let $\omega_1^f \in \Omega_{\tilde{x}_1}^1(X_1^f)$ and $\omega_2 \in \Omega_{\tilde{x}_2}^1(X_2)$ be 1-forms compatible with f_{\sim} . Consider $((\pi^f)^*)^{-1}(\omega_1^f, \omega_2) = \omega_1^f \cup_{f_{\sim}} \omega_2$.

Lemma 6.6. *The form $((\pi^f)^*)^{-1}(\omega_1^f, \omega_2)$ vanishes at x .*

Proof. Let $p : U \rightarrow X_1 \cup X_2$ be a plot centered at x ; assume U to be connected. Then p lifts to a plot p_i of X_i . Suppose it lifts to a plot p_2 of X_2 ; since the lift of x to X_2 is unique, it has to be \tilde{x}_2 , which implies that p_2 is centered at \tilde{x}_2 , and therefore

$$(((\pi^f)^*)^{-1}(\omega_1^f, \omega_2))(p)(0) = \omega_2(p_2)(0) = 0.$$

Assume now that p lifts to a plot p_1 of X_1 . Notice that $\pi^f \circ p_1$ is a plot of X_1^f , and it is centered at \tilde{x}_1 , since the lift of x to X_1^f . Thus, we have again

$$(((\pi^f)^*)^{-1}(\omega_1^f, \omega_2))(p)(0) = \omega_1^f(p_1)(0) = 0,$$

and the lemma is proven. \square

Corollary 6.7. *For x , \tilde{x}_1 , and \tilde{x}_2 as above, we have*

$$(\pi^f)^*(\Omega_x^1(X_1 \cup_f X_2)) = \Omega_{\tilde{x}_1}^1(X_1^f) \times_{comp} \Omega_{\tilde{x}_2}^1(X_2).$$

Let us now turn to the relation of the space $\Omega_{\tilde{x}_1}^1(X_1^f)$ to the subspaces of vanishing forms in $\Omega_f^1(X_1)$. Recall that we have already established the diffeomorphism of $\Omega_f^1(X_1)$ and $\Omega^1(X_1^f)$, so we are essentially asking, what becomes of subspaces of forms vanishing at a given point of X_1^f under this diffeomorphism (the pullback map $(\pi_1^f)^*$).

6.4.2 The pullback space $(\pi_1^f)^*(\Omega_y^1(X_1^f))$ for $y \in \pi_1^f(Y)$

Fix a point $y \in \pi_1^f(Y)$; let first $\omega \in \Omega_y^1(X_1^f)$ be a form vanishing at y . By the argument identical to that in the proof of Lemma 4.1, the pullback form $(\pi_1^f)^*(\omega) \in \Omega_f^1(X_1)$ vanishes at any lift of y . Indeed, if $\tilde{y} \in Y \subset X_1$ is such that $\pi_1^f(\tilde{y}) = y$, and $p : U \rightarrow X_1$ is a plot centered at \tilde{y} , then $\pi_1^f \circ p$ is a plot centered at y , and

$$((\pi_1^f)^*(\omega))(p)(0) = \omega(\pi_1^f \circ p)(0) = 0.$$

Thus, we obtain the following.

Proposition 6.8. *For any $y \in \pi_1^f(Y) \subset X_1^f$ we have*

$$(\pi_1^f)^*(\Omega_y^1(X_1^f)) = \cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_{\tilde{y}}^1(X_1)).$$

Proof. The inclusion $(\pi_1^f)^*(\Omega_y^1(X_1^f)) \subseteq \cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_{\tilde{y}}^1(X_1))$ has been proven immediately prior to the statement of the proposition, so it suffices to establish the reverse inclusion. This follows from the definition of the inverse of $(\pi_1^f)^*$. More precisely, suppose that $\omega_1 \in \cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_{\tilde{y}}^1(X_1))$; this means that ω_1 vanishes at every point $\tilde{y} \in X_1$ such that $\pi_1^f(\tilde{y}) = y$, which in turn means that for every plot p_1 centered at any such point we have $\omega_1(p_1)(0) = 0$. Let ω_1^f be the pushforward of the form ω_1 to X_1^f , that is, $\omega_1^f = ((\pi_1^f)^*)^{-1}(\omega_1)$; let p_1^f be any plot of X_1^f centered at y , and let p_1 be a lift of p_1^f to a plot of X_1 (such a lift exists by the definition of the pushforward diffeology), $p_1^f = \pi_1^f \circ p_1$. Notice that p_1^f is centered at some \tilde{y} such that $\pi_1^f(\tilde{y}) = y$, and ω_1 vanishes at all such points, therefore $\omega_1(p_1)(0) = 0$. Finally, recall that $\omega_1^f(p_1^f) = \omega_1(p_1)$ for any lift p_1 of the plot p_1^f . This allows us to conclude that $\omega_1^f(p_1^f)(0) = \omega_1(p_1)(0)$, and since p_1^f is arbitrary, we further conclude that ω_1^f vanishes at y , as we wanted. \square

6.4.3 The subspace $\left(\cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_{\tilde{y}}^1(X_1))\right) \times_{comp} \Omega_{\tilde{y}_2}^1(X_2)$

Here \tilde{y}_2 is the point of X_2 such that $f(\tilde{y}) = \tilde{y}_2$. The structure of the subspace in question depends on whether f is a subduction; we will assume that it is. Recall that, as has been established in the previous section, this ensures that the natural projections $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega_f^1(X_1)$ and $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$ are both surjective. We now need to see whether this holds for subspaces of vanishing forms; to avail ourselves of the tools used previously, we first consider the interaction between the vanishing of forms and the f -invariance. Here is what we mean.

Projection to $\Omega_{\tilde{y}_2}^1(X_2)$ Let us now use the above construction to show that the surjectivity of the projection is preserved for the subspaces of vanishing forms. We start from the second factor; as before, it is the easier case.

Proposition 6.9. *Let f be such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, let $y \in \pi_1^f(Y)$ be any point, and let $\tilde{y}_2 \in X_2 \cap (\pi^f)^{-1}(y)$. Then the projection $\left(\cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_{\tilde{y}}^1(X_1))\right) \times_{comp} \Omega_{\tilde{y}_2}^1(X_2) \rightarrow \Omega_{\tilde{y}_2}^1(X_2)$ is surjective.*

Proof. By Proposition 6.8 it suffices to show that $(\pi_1^f \times \text{Id}_{X_2})^*(\Omega_{\pi_1^f(y)}^1(X_1^f \cup_{f\sim} X_2))$ is a sub-direct product of $\Omega_y^1(X_1^f)$ and of $\Omega_{\tilde{y}_2}^1(X_2)$, and this follows from the assumptions and Proposition 5.3. \square

Projection to $\left(\cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_f^1)_{\tilde{y}}(X_1)\right)$ This has just been proven together with the case of the other factor.

Proposition 6.10. *Let f be a subduction such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, let $y \in \pi_1^f(Y)$ be any point, and let $\tilde{y}_2 \in X_2 \cap (\pi^f)^{-1}(y)$. Then the projection $\left(\cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_f^1)_{\tilde{y}}(X_1)\right) \times_{comp} \Omega_{\tilde{y}_2}^1(X_2) \rightarrow \left(\cap_{\tilde{y} \in (\pi_1^f)^{-1}(y)} (\Omega_f^1)_{\tilde{y}}(X_1)\right)$ is surjective.*

6.5 The pullbacks of the spaces of vanishing forms: summary

We collect here the final conclusions of this section regarding the image of the space $\Omega_x^1(X_1 \cup_f X_2)$ under the pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$.

Theorem 6.11. *Let f be a subduction such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, and let $x \in X_1 \cup_f X_2$. Then the following is true:*

1. *If $x \in i_1(X_1 \setminus Y)$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) = (\Omega_f^1)_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2).$$

2. *If $x \in i_2(X_2 \setminus f(Y))$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) = \Omega_f^1(X_1) \times_{comp} \Omega_{\pi^{-1}(x)}^1(X_2).$$

3. *If $x \in i_2(f(Y))$ then*

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) = \left(\cap_{\tilde{x} \in (\pi_1^f)^{-1}(x)} (\Omega_f^1)_{\tilde{x}}(X_1)\right) \times_{comp} \Omega_{\tilde{x}_2}^1(X_2),$$

where \tilde{x}_2 is such that $i_2(\tilde{x}_2) = x$.

7 The characteristic maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$

As a preliminary step towards a (relatively) detailed description of $\Lambda^1(X_1 \cup_f X_2)$, we define two maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$, each of which is defined on a subset of $\Lambda^1(X_1 \cup_f X_2)$ and takes values, respectively, in $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$.

7.1 The definition of $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$

By definition, the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ is obtained as the pseudo-bundle quotient of

$$(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2)),$$

considered as the trivial pseudo-bundle over $X_1 \cup_f X_2$, by its sub-bundle of vanishing 1-forms. Since $X_1 \cup_f X_2$ is a diffeological quotient of $X_1 \sqcup X_2$, $\Lambda^1(X_1 \cup_f X_2)$ is also a quotient of

$$\begin{aligned} (X_1 \sqcup X_2) \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2)) &\cong \\ &\cong (X_1 \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2))) \sqcup (X_2 \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2))). \end{aligned}$$

Let now

$$\begin{aligned} \rho_1 : X_1 \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2)) &\rightarrow X_1 \times \Omega^1(X_1), \\ \rho_2 : X_2 \times (\Omega^1(X_1) \times_{comp} \Omega^1(X_2)) &\rightarrow X_2 \times \Omega^1(X_2) \end{aligned}$$

be the maps acting by identity on X_1 or X_2 , whichever is relevant, and by the projection on either the first or the second factor on $\Omega^1(X_1) \times_{comp} \Omega^1(X_2)$.

Lemma 7.1. *The maps ρ_1 and ρ_2 descend to well-defined maps*

$$\tilde{\rho}_1^\Lambda : \Lambda^1(X_1 \cup_f X_2) \supset (\pi^\Lambda)^{-1}(i_1(X_1) \cup i_2(f(Y))) \rightarrow \Lambda^1(X_1) \quad \text{and} \quad \tilde{\rho}_2^\Lambda : (\pi^\Lambda)^{-1}(i_2(X_2)) \rightarrow \Lambda^1(X_2).$$

Proof. Consider first the map ρ_1 . It suffices to show that, for all $x \in i_1(X_1 \setminus Y)$, we have

$$\rho_1(\{x\} \times \pi^*(\Omega_x^1(X_1 \cup_f X_2))) \subseteq \{i_1^{-1}(x)\} \times \Omega_{i_1^{-1}(x)}^1(X_1),$$

and for all $x \in i_2(f(Y))$ we have

$$\rho_1(\{x\} \times \pi^*(\Omega_x^1(X_1 \cup_f X_2))) \subseteq \{f^{-1}(i_2^{-1}(x))\} \times \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1).$$

This is immediate from Theorem 6.11.

Likewise, for the map ρ_2 it is sufficient to prove that for $x \in i_2(X_2)$ there is the inclusion

$$\rho_2(\{x\} \times \pi^*(\Omega_x^1(X_1 \cup_f X_2))) \subseteq \{i_2^{-1}(x)\} \times \Omega_{i_2^{-1}(x)}^1 \times \Omega_{i_2^{-1}(x)}^1(X_2).$$

This is also an obvious consequence of Theorem 6.11. \square

7.2 The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are smooth

By Lemma 7.1 these maps are well-defined. We now need to show that they are smooth; since the two cases are symmetric, it suffices to consider one of them. Let us first explain why $\tilde{\rho}_1^\Lambda$ is smooth for the subset diffeology on its domain.

Proposition 7.2. *The map $\tilde{\rho}_1^\Lambda$ is smooth for the subset diffeology on $(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y) \cup i_2(f(Y))) \subseteq \Lambda^1(X_1 \cup_f X_2)$.*

Proof. Let $p : U \rightarrow \Lambda^1(X_1 \cup_f X_2)$ be a plot of $\Lambda^1(X_1 \cup_f X_2)$ such that for all $u \in U$ we have $\pi^\Lambda(p(u)) \in i_1(X_1 \setminus Y) \cup i_2(f(Y))$; we need to check that $\tilde{\rho}_1^\Lambda \circ p$ is a plot of $\Lambda^1(X_1)$. Since the diffeology on $\Lambda^1(X_1 \cup_f X_2)$ is the pushforward of the product diffeology on $(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$, we can assume that U is small enough so that p lifts to a pair of form (p_\cup, p^Ω) , where $p_\cup : U \rightarrow X_1 \cup_f X_2$ is a plot of $X_1 \cup_f X_2$ and $p^\Omega : U \rightarrow \Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$ is a plot of $\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$. The latter has essentially the product diffeology, therefore p^Ω , in turn, has form (p_1^Ω, p_2^Ω) , where each p_i^Ω is a plot of $\Omega^1(X_i)$.

It then remains to observe that $\rho_1 \circ (p_\cup, p^\Omega) = (\pi \circ p_\cup, p_1^\Omega)$, and therefore $\tilde{\rho}_1^\Lambda \circ p = \pi_1^{\Omega, \Lambda} \circ (\pi \circ p_\cup, p_1^\Omega)$. The right-hand side is by definition a plot of $\Lambda^1(X_1)$, so we obtain the desired claim. \square

Exactly the same statement, with a completely analogous proof, holds for the map $\tilde{\rho}_2^\Lambda$.

Proposition 7.3. *The map $\tilde{\rho}_2^\Lambda$ is smooth for the subset diffeology on $(\pi^\Lambda)^{-1}(i_2(X_2)) \subseteq \Lambda^1(X_1 \cup_f X_2)$.*

7.3 The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ characterize plots of $\Lambda^1(X_1 \cup_f X_2)$

Turning to consider the plots of $\Lambda^1(X_1 \cup_f X_2)$, we first observe that a map $p : U \rightarrow \Lambda^1(X_1 \cup_f X_2)$ with connected U is a plot only if its range is fully contained in the domain of definition of either $\tilde{\rho}_1^\Lambda$ or $\tilde{\rho}_2^\Lambda$. This is because for it to be a plot, it is necessary that $\pi^\Lambda \circ p$ be a plot of $X_1 \cup_f X_2$; for U connected that means that $\pi^\Lambda \circ p$ lifts to either a plot of X_1 or one of X_2 . This is in turn equivalent to the range of p being contained in the domain of definition of either $\tilde{\rho}_1^\Lambda$ or $\tilde{\rho}_2^\Lambda$. Accordingly, either $\tilde{\rho}_1^\Lambda \circ p$ or $\tilde{\rho}_2^\Lambda \circ p$ should be a plot of, respectively, $\Lambda^1(X_1)$ or $\Lambda^1(X_2)$. However, this is still a necessary condition and not a sufficient one in general. In this section we consider the assumption under which this does turn out to be a sufficient condition.

7.3.1 The two pushforward diffeologies \mathcal{D}_1^Ω and \mathcal{D}_2^Ω

Recall the natural inclusions $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$, and consider the corresponding pullback maps $i^* : \Omega^1(X_1) \rightarrow \Omega^1(Y)$ and $j^* : \Omega^1(X_2) \rightarrow \Omega^1(f(Y))$. In general, we cannot make a claim regarding such properties of these maps as surjectivity, and therefore we introduce the following assumption.

Denote by \mathcal{D}_1^Ω the diffeology on $\Omega^1(Y)$ that is the pushforward of the diffeology of $\Omega^1(X_1)$ by the map i^* . Since this map is smooth for the standard functional diffeology on $\Omega^1(Y)$, as all pullback maps are, \mathcal{D}_1^Ω is contained in this standard diffeology; notice that it may be properly contained. Next, let \mathcal{D}_2^Ω be another diffeology on $\Omega^1(Y)$, and precisely the one obtained as the pushforward of the standard functional diffeology on $\Omega^1(X_2)$ by the map f^*j^* . Also in this case, it is contained in the standard diffeology of $\Omega^1(Y)$, perhaps properly.

Our strongest assumption in what follows will be that

$$\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega.$$

It is satisfied in most standard contexts, such as those of connected simply-connected domains in \mathbb{R}^n , or its affine subsets.

7.3.2 The assumption $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ implies $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$

We have introduced so far two additional conditions on the gluing map f , expressed by the equalities mentioned in the title, that allow to obtain more complete statements regarding $\Omega^1(X_1 \cup_f X_2)$, and therefore, as we will see below, about $\Lambda^1(X_1 \cup_f X_2)$. We now show that of these two conditions, the former is the stronger one.

Proposition 7.4. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a gluing map. If $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ then $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$.*

Proof. Let $i^*(\omega_1) \in i^*(\Omega^1(X_1))$, where $\omega_1 \in \Omega^1(X_1)$ is an arbitrary element. Consider a constant map $p_1^\Omega : U \rightarrow \{\omega_1\} \subset \Omega^1(X_1)$; this is a plot of $\Omega^1(X_1)$ since all constant maps are so. By assumption, $i^* \circ p_1^\Omega \in \mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Since \mathcal{D}_2^Ω is defined as the pushforward of the diffeology of $\Omega^1(X_2)$ by the map f^*j^* , there exists a plot $p_2^\Omega : U \rightarrow \Omega^1(X_2)$ of $\Omega^1(X_2)$ such that $i^* \circ p_1^\Omega = (f^*j^*) \circ p_2^\Omega$. Let $\omega_2 \in \Omega^1(X_2)$ be any form in the range of p_2^Ω . Then $i^*(\omega_1) = (f^*j^*)(\omega_2)$ for any arbitrary $\omega_1 \in \Omega^1(X_1)$, which means that $i^*(\Omega^1(X_1)) \subseteq (f^*j^*)(\Omega^1(X_2))$. The reverse inclusion is proved in exactly the same way, therefore the claim. \square

7.3.3 The assumption $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ allows to characterize the diffeology of $\Lambda^1(X_1 \cup_f X_2)$

One of the reasons why we consider the assumption $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ is that it allows to make the following observation regarding the diffeology of $\Lambda^1(X_1 \cup_f X_2)$.

Theorem 7.5. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supset Y \rightarrow X_2$ be a diffeomorphism of its domain with its image such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Then the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ is the coarsest diffeology such that both maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are smooth.*

Proof. Let \mathcal{D}' be any diffeology on $\Lambda^1(X_1 \cup_f X_2)$ such that $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are smooth, and let $s : U' \rightarrow \Lambda^1(X_1 \cup_f X_2)$ be a plot of \mathcal{D}' . It suffices to show that for every $u' \in U'$ there is a neighborhood U of u' such that $s|_U$ lifts to a plot of $(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$.

Assume, as we always can, that U' is connected. Then $\pi^\Lambda \circ s$, which is a plot of $X_1 \cup_f X_2$, lifts to either a plot s_1 of X_1 or to a plot s_2 of X_2 . Suppose that $\pi^\Lambda \circ s$ lifts to a plot s_1 of X_1 . Then $\tilde{\rho}_1^\Lambda \circ s$ is a plot of $\Omega^1(X_1)$ by assumption. We therefore can assume that U' is small enough so that it lifts to a plot of $X_1 \times \Omega^1(X_1)$ and, furthermore, that this lift has form (s'_1, s_1^Ω) for a plot s'_1 of X_1 and a plot s_1^Ω of $\Omega^1(X_1)$. Then obviously s'_1 coincides with s_1 whenever both are defined. We thus have

$$\tilde{\rho}_1^\Lambda \circ s = \pi_1^{\Omega, \Lambda} \circ (s_1, s_1^\Omega),$$

where $\pi_1^{\Omega, \Lambda} : X_1 \times \Omega^1(X_1) \rightarrow \Lambda^1(X_1)$ is the defining projection.

Consider now $i^* \circ s_1^\Omega$; this is a plot of $\Omega^1(Y)$, which is contained in the diffeology \mathcal{D}_1^Ω . Then by assumption it belongs to \mathcal{D}_2^Ω as well, that is, there is a plot $s_2^\Omega : U \rightarrow \Omega^1(X_2)$ such that $i^* \circ s_1^\Omega = f^* \circ j^* \circ s_2^\Omega$; in particular, $s_1^\Omega(u)$ and $s_2^\Omega(u)$ are always compatible. It now suffices to observe that the triple $(\pi^\Lambda \circ s, (s_1^\Omega, s_2^\Omega))$ is a plot of $(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$ and that by construction it covers s , that is,

$$s = \pi^{\Omega, \Lambda} \circ (\pi^\Lambda \circ s, (s_1^\Omega, s_2^\Omega)).$$

Since the case when $\pi^\Lambda \circ s$ is fully analogous, this proves the claim. \square

8 The structure of the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$

We now complete our description of the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ of the values of differential 1-forms on $X_1 \cup_f X_2$. Do notice that our results require the same assumptions of f being a diffeomorphism and such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$, although in some cases the weaker assumption of $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ will suffice. The main tool is that of the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$, so we first consider other relevant properties of them.

8.1 The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$, and the fibres of $\Lambda^1(X_1 \cup_f X_2)$

We consider first is the potential surjectivity/bijection of these maps.

8.1.1 The restrictions of $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ over $i_1(X_1 \setminus Y)$ and $i_2(X_2 \setminus f(Y))$ are fibrewise bijections

We first show that, as long as the condition $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ is satisfied, the restrictions of the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ to fibres over the points outside of the domain of gluing are bijections.

Proposition 8.1. *Let X_1 and X_2 be two diffeological spaces, let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, and let $x \in i_1(X_1 \setminus Y)$. Then the restriction*

$$\tilde{\rho}_1^\Lambda|_{\Lambda_x^1(X_1 \cup_f X_2)} : \Lambda_x^1(X_1 \cup_f X_2) \rightarrow \Lambda_{i_1^{-1}(x)}^1(X_1)$$

is a bijection.

Proof. By definition, any element α of the fibre $\Lambda_x^1(X_1 \cup_f X_2)$ has form

$$\alpha = (\omega_1, \omega_2) + \Omega_x^1(X_1 \cup_f X_2),$$

where $\omega_1 \in \Omega^1(X_1)$ and $\omega_2 \in \Omega^1(X_2)$ are compatible forms. By Theorem 6.11, we then have

$$\alpha = (\omega_1, \omega_2) + \Omega_{i_1^{-1}(x)}^1(X_1) \times_{\text{comp}} \Omega^1(X_2),$$

where we freely identify $\Omega^1(X_1 \cup_f X_2)$ with its image $\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$ under the pullback map π^* . The map $\tilde{\rho}_1^\Lambda$ then acts by the assignment

$$\alpha \mapsto \tilde{\rho}_1^\Lambda(\alpha) = \omega_1 + \Omega_{i_1^{-1}(x)}^1(X_1),$$

and it suffices to show that it has an inverse. We define this prospective inverse as follows.

Let $\omega_1 \in \Omega^1(X_1)$ be any form, and let $\alpha_1 := \omega_1 + \Omega_{i_1^{-1}(x)}^1(X_1)$. By the assumption that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ there always exists at least one form $\omega_2 \in \Omega^1(X_2)$ such that ω_1 and ω_2 are compatible. We define

$$(\tilde{\rho}_1^\Lambda)^{-1}(\alpha_1) := (\omega_1, \omega_2) + \Omega_{i_1^{-1}(x)}^1(X_1) \times_{\text{comp}} \Omega^1(X_2);$$

it now suffices to show that this element does not depend on the choice of ω_2 .

Let $\omega'_2 \in \Omega^1(X_2)$ be another form compatible with ω_1 ; we need to check that $\omega_2 - \omega'_2 \in \Omega_{i_1^{-1}(x)}^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$. Since $(f^*j^*)(\omega_2 - \omega'_2) = i^*(\omega_1) - i^*(\omega_1) = 0$ by assumption, the form $\omega_2 - \omega'_2$ is compatible with the zero form, whence the desired conclusion, which completes the proof. \square

The analogous statement, with a completely similar proof (which we therefore omit), is also true for the other factor of the gluing.

Proposition 8.2. *Let X_1 and X_2 be two diffeological spaces, let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, and let $x \in i_2(X_2 \setminus f(Y))$. Then the restriction*

$$\tilde{\rho}_2^\Lambda|_{\Lambda_x^1(X_1 \cup_f X_2)} : \Lambda_x^1(X_1 \cup_f X_2) \rightarrow \Lambda_{i_2^{-1}(x)}^1(X_2)$$

is a bijection.

8.1.2 The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are surjective if and only if $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$

We have just seen that this is true in one direction on fibres over points outside of the domain of gluing. It remains to check that the condition $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ is actually an equivalence, and that the entire statement holds for points $x \in i_2(f(Y))$.

Proposition 8.3. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a gluing diffeomorphism. Then:*

1. *Let f be such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, and let $x \in i_2(f(Y))$. Then the restrictions*

$$\tilde{\rho}_1^\Lambda|_{\Lambda_x^1(X_1 \cup_f X_2)} : \Lambda_x^1(X_1 \cup_f X_2) \rightarrow \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \quad \text{and} \quad \tilde{\rho}_2^\Lambda|_{\Lambda_x^1(X_1 \cup_f X_2)} : \Lambda_x^1(X_1 \cup_f X_2) \rightarrow \Lambda_{i_2^{-1}(x)}^1(X_2)$$

are surjective;

2. *Let $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ be surjective as maps into $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. Then $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$.*

Proof. Let us prove 1. Let $x \in i_2(f(Y))$, and let $\alpha \in \Lambda_x^1(X_1 \cup_f X_2)$. By Theorem 6.11 and by the assumption that f is a diffeomorphism we have

$$\alpha = (\omega_1, \omega_2) + \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1) \times_{\text{comp}} \Omega_{i_2^{-1}(x)}^1(X_2).$$

The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ act by

$$\tilde{\rho}_1^\Lambda(\alpha) = \omega_1 + \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1) \quad \text{and} \quad \tilde{\rho}_2^\Lambda(\alpha) = \omega_2 + \Omega_{i_2^{-1}(x)}^1(X_2).$$

Let us show, for instance, that $\tilde{\rho}_1^\Lambda$ is surjective. Let $\omega_1 + \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1)$ be any element of $\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1)$. The assumption that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ means that there exists at least one form $\omega_2 \in \Omega^1(X_2)$ compatible with ω_1 . The element

$$\alpha := (\omega_1, \omega_2) + \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1) \times_{\text{comp}} \Omega_{i_2^{-1}(x)}^1(X_2)$$

is then such that $\tilde{\rho}_1^\Lambda(\alpha) = \omega_1 + \Omega_{f^{-1}(i_2^{-1}(x))}^1(X_1)$; since the latter is an arbitrary element of $\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1)$, we conclude that $\tilde{\rho}_1^\Lambda$ is surjective on the fibre in question. In the case of $\tilde{\rho}_2^\Lambda$ the proof is analogous.

Let us now prove 2. Let $\omega_1 \in \Omega^1(X_1)$; we need to show that there exists a form $\omega_2 \in \Omega^1(X_2)$ such that $i^*(\omega_1) = (f^*j^*)(\omega_2)$, equivalently, such that ω_1 and ω_2 are compatible. Let $x \in X_1$ be an arbitrary point; to give a uniform treatment we assume that $x \in Y$. Since by assumption $\tilde{\rho}_1^\Lambda$ is surjective, there exists $\alpha \in \Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2)$ such that $\tilde{\rho}_1^\Lambda(\alpha) = \pi_1^{\Omega, \Lambda}(x, \omega_1)$, the element of $\Lambda_x^1(X_1)$ determined by ω_1 . Let (ω_1, ω_2) be any pair in $(\pi^{\Omega, \Lambda})^{-1}(\alpha)$; then ω_1 and ω_2 are compatible by construction, which yields the claim. \square

8.1.3 The restrictions of $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ over $i_1(X_1 \setminus Y)$ and $i_2(X_2 \setminus f(Y))$ are diffeomorphisms if $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$

Propositions 8.1 and 8.2 above show that any fibre of $\Lambda^1(X_1 \cup_f X_2)$ over a point that is not in the domain of gluing, there is a fibrewise isomorphism, which is furthermore smooth, with the appropriate fibre of either $\Lambda^1(X_1)$ or $\Lambda^1(X_2)$. Indeed, these isomorphisms are given by the restrictions of the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$. This yields smooth pseudo-bundle isomorphisms

$$\tilde{\rho}_1^\Lambda|_{(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))} : (\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y)) \rightarrow (\pi_1^\Lambda)^{-1}(X_1 \setminus Y) \text{ and}$$

$$\tilde{\rho}_2^\Lambda|_{(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))} : (\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y))) \rightarrow (\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y)).$$

covering i_1^{-1} and $i_2^{-1}|_{i_2(X_2 \setminus f(Y))}$. We now consider the conditions under which they have smooth inverses.

Proposition 8.4. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a gluing diffeomorphism. The maps $\tilde{\rho}_1^\Lambda|_{(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))}$ and $\tilde{\rho}_2^\Lambda|_{(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))}$ have smooth, for the subset diffeologies on their domains and ranges, inverses if $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$.*

Proof. Suppose first that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Let $p_1 : U \rightarrow \Lambda^1(X_1)$ be a plot of $\Lambda^1(X_1)$ such that the range of $\pi_1^\Lambda \circ p_1$ is contained in $X_1 \setminus Y$. We need to show that, up to appropriately restricting U , there exists a plot p of $\Lambda^1(X_1 \cup_f X_2)$ such that $p_1 = \tilde{\rho}_1^\Lambda \circ p$. Since the diffeology of $\Lambda^1(X_1)$ is the pushforward of the diffeology of $X_1 \times \Omega^1(X_1)$ (by the map $\pi_1^{\Omega, \Lambda}$), we can assume that U is also small enough so that p_1 has a lift, to a plot of $X_1 \times \Omega^1(X_1)$, of form $(\pi_1^\Lambda \circ p_1, p_1^\Omega)$, where p_1^Ω is a plot of $\Omega^1(X_1)$.

By the assumption that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ there exists a plot p_2^Ω of $\Omega^1(X_2)$ such that $i^* \circ p_1^\Omega = (f^* j^*) \circ p_2^\Omega$. This means that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$, therefore $(i_1 \circ (\pi_1^\Lambda \circ p_1), (p_1^\Omega, p_2^\Omega))$ is a plot of $(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$, therefore its composition

$$\pi^{\Omega, \Lambda} \circ (i_1 \circ (\pi_1^\Lambda \circ p_1), (p_1^\Omega, p_2^\Omega))$$

is the desired plot p . The case of $\tilde{\rho}_2^\Lambda$ is analogous. \square

Remark 8.5. *It is not entirely clear to us at the moment whether the condition $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ is a necessary one. Indeed, assuming that $(\tilde{\rho}_1^\Lambda|_{(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))})^{-1}$ and $(\tilde{\rho}_2^\Lambda|_{(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))})^{-1}$ are smooth, a standard reasoning allows us to conclude that for every point $x_1 \in X_1 \setminus Y$, up to choosing a smaller U , there exist plots q_1^Ω and q_2^Ω are plots of $\Omega^1(X_1)$ and $\Omega^1(X_2)$ respectively such that $q_1^\Omega(u)$ and $q_2^\Omega(u)$ are compatible for all $u \in U$, and*

$$p_1^\Lambda(u) = \pi_1^{\Omega, \Lambda}(x_1, p_1^\Omega(u)) = \pi_1^{\Omega, \Lambda}(x_1, q_1^\Omega(u)).$$

But since vanishing of forms does not have a strict correlation with compatibility, it is not clear why this should imply the (pointwise) compatibility of p_1^Ω with q_2^Ω . We leave this question unanswered, since the above statement is sufficient for our purposes.

8.1.4 Compatibility of elements in $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, and the pullback map f_Λ^*

To proceed, we need a certain compatibility notion for elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$. Two such elements are called compatible if they are images, under the maps $\pi_1^{\Omega, \Lambda}$ and $\pi_2^{\Omega, \Lambda}$ respectively, of two elements of form (y, ω_1) and $(f(y), \omega_2)$, where $y \in Y$ and ω_1 and ω_2 are compatible as forms in $\Omega^1(X_1)$ and $\Omega^1(X_2)$. A more direct way to state this definition is the following one (recall that each $\alpha_i \in \Lambda^1(X_i)$ is a coset of form $\omega_i + \Omega_{x_i}^1(X_i)$).

Definition 8.6. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a smooth map. Two elements $\alpha_1 \in \Lambda^1(X_1)$ and $\alpha_2 \in \Lambda^1(X_2)$ are said to be **compatible** if for any $\omega_1 \in \alpha_1$ and $\omega_2 \in \alpha_2$ the forms ω_1 and ω_2 are compatible.*

The definition as stated is applicable to any smooth map. In the case when f is at least a diffeomorphism, it is possible to define the corresponding pullback map $f_\Lambda^* : \Lambda^1(f(Y)) \rightarrow \Lambda^1(Y)$ and use it to characterize pairs of compatible α_1, α_2 . Notice that f_Λ^* is not defined in general between the entire

pseudo-bundles $\Lambda^1(X_2)$ and $\Lambda^1(X_1)$, nor are its domain and its range $\Lambda^1(f(Y))$ and $\Lambda^1(Y)$ sub-bundles in $\Lambda^1(X_2)$ and $\Lambda^1(X_1)$ (the proof of this and the other statements appearing in this section can be found in [9]).

The pullback map f_Λ^* is the pushforward of the map $(f^{-1}, f^*) : f(Y) \times \Omega^1(f(Y)) \rightarrow Y \times \Omega^1(Y)$ that acts by

$$(f^{-1}, f^*)(f(y), \omega_2) = (y, f^*\omega_2),$$

by the defining projections

$$\pi_{f(Y)}^{\Omega, \Lambda} : f(Y) \times \Omega^1(f(Y)) \rightarrow \Lambda^1(f(Y)) \quad \text{and} \quad \pi_Y^{\Omega, \Lambda} : Y \times \Omega^1(Y) \rightarrow \Lambda^1(Y)$$

of $\Lambda^1(f(Y))$ and $\Lambda^1(Y)$ respectively. We remark that the following then holds:

$$\pi_Y^{\Omega, \Lambda} \circ (f^{-1}, f^*) = f_\Lambda^* \circ \pi_{f(Y)}^{\Omega, \Lambda}.$$

In order to apply f_Λ^* for obtaining a criterion of compatibility for elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, we need the following statement.

Proposition 8.7. (Lemma 5.2, [9]) *Let $Y' \subseteq X'$ be an arbitrary subspace (i.e., any subset endowed with the subset diffeology) of a diffeological space X' , and let $i : Y' \hookrightarrow X'$ be the natural inclusion map. The map $(i^{-1}, i^*) : i(Y') \times \Omega^1(X') \rightarrow Y' \times \Omega^1(Y')$ descends to a well-defined map*

$$i_\Lambda^* : \Lambda^1(X') \supseteq (\pi^\Lambda)^{-1}(Y') \rightarrow \Lambda^1(Y') \quad \text{such that} \quad \pi_{Y'}^{\Omega, \Lambda} \circ (i^{-1}, i^*) = i_\Lambda^* \circ \pi_{i(Y') \times \Omega^1(X')}^{\Omega, \Lambda}.$$

Applying this statement to $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$ yields the maps $i_\Lambda^* : \Lambda^1(X_1) \supseteq (\pi_1^\Lambda)^{-1}(Y) \rightarrow \Lambda^1(Y)$ such that $\pi_Y^{\Omega, \Lambda} \circ (i^{-1}, i^*) = i_\Lambda^* \circ \pi_{i(Y) \times \Omega^1(X_1)}^{\Omega, \Lambda}$ and $j_\Lambda^* : \Lambda^1(X_2) \supseteq (\pi_2^\Lambda)^{-1}(f(Y)) \rightarrow \Lambda^1(f(Y))$ such that $\pi_{f(Y)}^{\Omega, \Lambda} \circ (j^{-1}, j^*) = j_\Lambda^* \circ \pi_{j(f(Y)) \times \Omega^1(X_2)}^{\Omega, \Lambda}$, and ultimately leads to the following statement.

Proposition 8.8. (Proposition 5.4, [9]) *Two elements $\alpha_1 \in \Lambda^1(X_1)$ and $\alpha_2 \in \Lambda^1(X_2)$ are compatible if and only if $\pi_1^\Lambda(\alpha_1) \in Y$, $\pi_2^\Lambda(\alpha_2) = f(\pi_1^\Lambda(\alpha_1))$, and*

$$i_\Lambda^* \alpha_1 = f^*(j_\Lambda^* \alpha_2).$$

8.1.5 The map $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is bijective on each fibre over the domain of gluing

In this and the following subsections we consider the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ on fibres over the domain of gluing, i.e., over points in $i_2(f(Y))$. On such fibres, both $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are defined, and furthermore their images are always compatible. It therefore makes sense to speak of the map

$$\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda : (\pi^\Lambda)^{-1}(i_2(f(Y))) \rightarrow (\pi_1^\Lambda)^{-1}(Y) \oplus_{\text{comp}} (\pi_2^\Lambda)^{-1}(f(Y)),$$

where $(\pi_1^\Lambda)^{-1}(Y) \oplus_{\text{comp}} (\pi_2^\Lambda)^{-1}(f(Y))$ is the sub-bundle of the direct sum pseudo-bundle $(\pi_1^\Lambda)^{-1}(Y) \oplus (\pi_2^\Lambda)^{-1}(f(Y))$ whose fibres consist precisely of pairs of compatible elements in $\Lambda_y^1(X_1)$ and $\Lambda_{f(y)}^1(X_2)$, for all $y \in Y$ (the action of the map itself should then be obvious).

Proposition 8.9. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism with its image. Let $y \in Y$ be any point. Then*

$$\Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2) = (\Omega^1(X_1) \oplus_{\text{comp}} \Omega^1(X_2)) / \left(\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2) \right)$$

is diffeomorphic to $\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2)$ via the appropriate restriction of the map $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$.

Proof. Let $(\omega_1, \omega_2) + \left(\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2) \right)$ be a coset in

$$(\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) / \left(\Omega_y^1(X_1) \times_{\text{comp}} \Omega_{f(y)}^1(X_2) \right) \cong \Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2).$$

Its image under the map $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is the pair $(\omega_1 + \Omega_y^1(X_1), \omega_2 + \Omega_{f(y)}^1(X_2))$. Let now (ω_1, ω_2) be a representative of an element of $(\pi_1^\Lambda)^{-1}(y) \oplus_{\text{comp}} (\pi_2^\Lambda)^{-1}(f(y))$, that is, of a pair $(\omega_1 + \Omega_y^1(X_1), \omega_2 + \Omega_{f(y)}^1(X_2))$ such that every form in $\omega_1 + \Omega_y^1(X_1)$ is compatible with every form in $\omega_2 + \Omega_{f(y)}^1(X_2)$. It suffices to show that $(\omega_1, \omega_2) + (\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2))$ is a well-defined element of

$$(\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) / (\Omega_y^1(X_1) \times_{\text{comp}} \Omega_{f(y)}^1(X_2)) \cong \Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2),$$

that is, if $\omega'_1 \in \omega_1 + \Omega_y^1(X_1)$ and $\omega'_2 \in \omega_2 + \Omega_{f(y)}^1(X_2)$ then

$$(\omega'_1, \omega'_2) + (\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2)) = (\omega_1, \omega_2) + (\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2)).$$

This is equivalent to $\omega_1 - \omega'_1$ being compatible with $\omega_2 - \omega'_2$. Since by the definition of compatibility both ω_1 and ω'_1 are compatible with any form in $\omega_2 + \Omega_{f(y)}^1(X_2)$, we obtain the desired conclusion.

Thus, the assignment

$$(\omega_1 + \Omega_y^1(X_1), \omega_2 + \Omega_{f(y)}^1(X_2)) \mapsto (\omega_1, \omega_2) + (\Omega_y^1(X_1) \oplus_{\text{comp}} \Omega_{f(y)}^1(X_2))$$

is a well-defined inverse of $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda|_{\Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2)}$, and it only remains to show that it is smooth. Let $p^{\Lambda, \oplus} : U \rightarrow \Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2)$ be a plot of $\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2)$. Since the diffeology of the latter is a subset diffeology relative to a product diffeology, locally it has form $(p_1^\Lambda, p_2^\Lambda)$, where p_1^Λ is a plot of $\Lambda_y^1(X_1)$, p_2^Λ is a plot of $\Lambda_{f(y)}^1(X_2)$, and $p_1^\Lambda(u)$ and $p_2^\Lambda(u)$ are compatible for all $u \in U$. Notice that there exist plots p_1^Ω of $\omega_1 + \Omega_y^1(X_1)$ and p_2^Ω of $\omega_2 + \Omega_{f(y)}^1(X_2)$ such that $p_1^\Lambda(u) = \pi_1^{\Omega, \Lambda}(y, p_1^\Omega(u))$ and $p_2^\Lambda(u) = \pi_2^{\Omega, \Lambda}(f(y), p_2^\Omega(u))$. Furthermore, since by definition every form in $\omega_1 + \Omega_y^1(X_1)$ is compatible with every form in $\omega_2 + \Omega_{f(y)}^1(X_2)$, $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$. Therefore $p : U \rightarrow \Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2)$ given by

$$p(u) = \pi^{\Omega, \Lambda}(i_2(f(y)), (p_1^\Omega(u), p_2^\Omega(u)))$$

is a plot of $\Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2)$ such that

$$p^{\Lambda, \oplus} = (\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda) \circ p,$$

as wanted. \square

Remark 8.10. Notice that this statement does not require either one of our two usual assumptions of $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ or $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. This is because in this case the compatibility is accounted for in both the domain and the range of the map $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$.

8.1.6 The map $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is a diffeomorphism on fibres over $i_2(f(Y))$ if $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$

It follows from Proposition 8.9 that $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is a diffeomorphism on each fibre. It is furthermore smooth across the fibres in its domain of definition, that is, over the domain of gluing, because both maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are individually so. We now show that it is smoothly invertible across the fibres over the entire domain of gluing.

Proposition 8.11. Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism with its image. Then $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is a diffeomorphism

$$\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda : (\pi^\Lambda)^{-1}(i_2(f(Y))) \rightarrow (\pi_1^\Lambda)^{-1}(Y) \oplus_{\text{comp}} (\pi_2^\Lambda)^{-1}(f(Y)).$$

Proof. In view of Proposition 8.9, it suffices to show that the inverse of $\tilde{\rho}_1^\Lambda \oplus_{\text{comp}} \tilde{\rho}_2^\Lambda$ is smooth. The proof is exactly the same as that of smoothness within a single fibre in the concluding part of Proposition 8.9, so we omit it. \square

One extra remark can be made now.

Corollary 8.12. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a gluing diffeomorphism. Then:*

1. *The map*

$$\tilde{\rho}_1^\Lambda : \Lambda^1(X_1 \cup_f X_2) \supseteq (\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y) \cup i_2(f(Y))) \rightarrow \Lambda^1(X_1)$$

is a subduction onto its range if $\mathcal{D}_1^\Omega \subseteq \mathcal{D}_2^\Omega$. In particular, $\tilde{\rho}_1^\Lambda$ is a subduction onto $\Lambda^1(X_1)$ if and only if in addition we have that $i^(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$;*

2. *The map*

$$\tilde{\rho}_2^\Lambda : \Lambda^1(X_1 \cup_f X_2) \supseteq (\pi^\Lambda)^{-1}(i_2(X_2)) \rightarrow \Lambda^1(X_2)$$

is a subduction onto its range if $\mathcal{D}_2^\Omega \subseteq \mathcal{D}_1^\Omega$. In particular, $\tilde{\rho}_2^\Lambda$ is a subduction onto $\Lambda^1(X_2)$ if and only if we additionally have that $i^(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$.*

8.2 The fibres of $\Lambda^1(X_1 \cup_f X_2)$ and its natural decomposition

The following statement is a summary of Propositions 8.1, 8.2, and 8.9, and of Propositions 8.4 and 8.11.

Theorem 8.13. *Let X_1 and X_2 be two diffeological spaces, let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism with its image such that $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$, and let $x \in X_1 \cup_f X_2$ be a point. Then*

1. *If $x \in i_1(X_1 \setminus Y)$ then*

$$\Lambda_x^1(X_1 \cup_f X_2) \cong \Lambda_{i_1^{-1}(x)}^1(X_1).$$

2. *If $x \in i_2(X_2 \setminus f(Y))$ then*

$$\Lambda_x^1(X_1 \cup_f X_2) \cong \Lambda_{i_2^{-1}(x)}^1(X_2).$$

3. *If $x \in i_2(f(Y))$ then*

$$\Lambda_x^1(X_1 \cup_f X_2) \cong \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \times_{comp} \Lambda_{i_2^{-1}(x)}^1(X_2).$$

If furthermore f is such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ then the following three maps are diffeomorphisms for the appropriate subset diffeologies on their domains and their ranges:

$$1. \tilde{\rho}_1^\Lambda|_{(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))} : (\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y)) \rightarrow (\pi_1^\Lambda)^{-1}(X_1 \setminus Y);$$

$$2. \tilde{\rho}_1^\Lambda \oplus_{comp} \tilde{\rho}_2^\Lambda : (\pi^\Lambda)^{-1}(i_2(f(Y))) \rightarrow (\pi_1^\Lambda)^{-1}(Y) \oplus_{comp} (\pi_2^\Lambda)^{-1}(f(Y));$$

$$3. \tilde{\rho}_2^\Lambda|_{(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))} : (\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y))) \rightarrow (\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y)).$$

As we have mentioned before, the conditions $i^*(\Omega^1(X_1)) = (f^*j^*)(\Omega^1(X_2))$ and $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ are/might be too strong sometimes. But they are satisfied often enough, so we leave the statement in the form just given.

8.3 On simple examples

We briefly comment on what becomes of the above constructions when they are applied to the standard Euclidean spaces. A usual 1-form $\omega = \sum_{i=1}^n f_i(x)dx^i$ on \mathbb{R}^n is both an element of the vector space $\Omega^1(\mathbb{R}^n)$, assigning to any usual smooth function $p : U \rightarrow \mathbb{R}^n$ the usual form $p^*\omega$ on U . It also defined a smooth section of $\Lambda^1(\mathbb{R}^n)$, whose value at a given point $x_0 \in \mathbb{R}^n$ is the linear combination $\sum_{i=1}^n f_i(x_0)dx^i$.

To illustrate compatibility, consider a gluing of two copies of the standard \mathbb{R}^2 along the identity map between their y -axes. In this case two forms $g_1(x, y)dx + g_2(x, y)dy$ and $h_1(x, y)dx + h_2(x, y)dy$ are compatible if and only if $g_2(0, y) = h_2(0, y)$ for all $y \in \mathbb{R}$. Finally, in the case of one-point gluing (a wedge of two spaces) the compatibility does not provide any restriction on compatibility of elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, due to the fact that all forms vanish on constant plots. Thus, if x is the wedge point then

$$\Lambda_x^1(X_1 \vee X_2) \cong \Lambda_x^1(X_1) \oplus \Lambda_x^1(X_2).$$

9 Existence of a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$

In this section we consider the existence of an induced pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$, under the assumption that both $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ admit pseudo-metrics,¹² and that f is a diffeomorphism such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$ (although this may not be a necessary assumption). Since the latter is not the result of a gluing of the former two together, we cannot apply the gluing construction for the pseudo-metrics either. However, we do something similar and obtain one on $\Lambda^1(X_1 \cup_f X_2)$ by combining the given two; this requires additional assumptions on them.

9.1 The compatibility of pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$

It is intuitively clear that it is not possible to get a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$ out of just any two arbitrary pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$. We need a certain compatibility notion; the most natural one is the following.

Definition 9.1. *Let g_1^Λ and g_2^Λ be pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. We say that g_1^Λ and g_2^Λ are **compatible**, if for any $y \in Y$ and for any two compatible pairs (ω', ω'') and (μ', μ'') , where $\omega', \mu' \in \Lambda^1(X_1)$ and $\omega'', \mu'' \in \Lambda^1(X_2)$, we have*

$$g_1^\Lambda(y)(\omega', \mu') = g_2^\Lambda(f(y))(\omega'', \mu'').$$

9.2 The definition of the induced pseudo-metric g^Λ

We now define the induced pseudo-metric g^Λ on $\Lambda^1(X_1 \cup_f X_2)$. Let g_1^Λ and g_2^Λ be two compatible pseudo-metrics, on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. The map $g^\Lambda : X_1 \cup_f X_2 \rightarrow (\Lambda^1(X_1 \cup_f X_2))^* \otimes (\Lambda^1(X_1 \cup_f X_2))^*$ is defined as follows:

$$g^\Lambda(x)(\cdot, \cdot) = \begin{cases} g_1^\Lambda(i_1^{-1}(x))(\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)), & \text{if } x \in i_1(X_1 \setminus Y), \\ g_2^\Lambda(i_2^{-1}(x))(\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot)), & \text{if } x \in i_2(X_2 \setminus f(Y)), \\ \frac{1}{2} (g_1^\Lambda(f^{-1}(i_2^{-1}(x))) (\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)) + g_2^\Lambda(i_2^{-1}(x)) (\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot))), & \text{if } x \in i_2(f(Y)). \end{cases}$$

In the section immediately following, we show that this is indeed a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$, that is, that it has the correct rank on each fibre, and that it is smooth.

9.3 Proving that g^Λ is a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$

It is clear from the construction that there are two items to be checked: one, that g^Λ yields a pseudo-metric on each individual fibre, that is, that it has the maximal rank possible, and two, that it is smooth.

The rank of g^Λ We first check that g^Λ yields pseudo-metrics on individual fibres. By construction, $g^\Lambda(x)$ for $x \in X_1 \cup_f X_2$ is always a smooth symmetric bilinear form on $\Lambda_x^1(X_1 \cup_f X_2)$. Therefore it suffices to show that it has the maximal possible rank on each fibre; this is the content of the following statement.

Lemma 9.2. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a gluing diffeomorphism such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Then for any $x \in X_1 \cup_f X_2$, the rank of g^Λ is equal to $\dim((\Lambda_x^1(X_1 \cup_f X_2))^*)$.*

Proof. Over points in $i_1(X_1 \setminus Y)$ and those in $i_2(X_2 \setminus f(Y))$ this follows directly from the construction. Let $x \in i_2(f(Y))$. Recall that for any such x the fibre $\Lambda_x^1(X_1 \cup_f X_2)$ then has form $\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus_{\text{comp}} \Lambda_{i_2^{-1}(x)}^1(X_2)$, which is a subspace in $\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus \Lambda_{i_2^{-1}(x)}^1(X_2)$. The definition of g^Λ obviously extends to that of a symmetric bilinear form on the latter space, and furthermore, this form is proportional to the usual direct sum form¹³ $g_1^\Lambda(f^{-1}(i_2^{-1}(x))) + g_2^\Lambda(i_2^{-1}(x))$, which obviously has the maximal rank

¹²Notice that this assumption includes that of these pseudo-bundles having only finite-dimensional fibres, which they do, for instance, if X_1 and X_2 are subsets of a standard \mathbb{R}^n .

¹³Defined by requiring the two factors to be orthogonal, and the restriction to each factor to coincide with the given form g_i^Λ .

possible. Therefore $g^\Lambda(x)$, being a restriction of $g_1^\Lambda(f^{-1}(i_2^{-1}(x))) + g_2^\Lambda(i_2^{-1}(x))$ to the vector subspace $\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus_{\text{comp}} \Lambda_{i_2^{-1}(x)}^1(X_2)$, has the maximal rank as well. \square

The smoothness of g^Λ It remains to show that g^Λ is smooth as a map

$$X_1 \cup_f X_2 \rightarrow (\Lambda^1(X_1 \cup_f X_2))^* \otimes (\Lambda^1(X_1 \cup_f X_2))^* ;$$

the proof of this will yield the statement that follows.

Theorem 9.3. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. The map $g^\Lambda : X_1 \cup_f X_2 \rightarrow (\Lambda^1(X_1 \cup_f X_2))^* \otimes (\Lambda^1(X_1 \cup_f X_2))^*$ is a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$.*

Proof. It is sufficient to choose plots $p : U \rightarrow X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ and $q, s : U' \rightarrow \Lambda^1(X_1 \cup_f X_2)$, and to show that the evaluation map

$$(u, u') \mapsto g^\Lambda(p(u))(q(u'), s(u')),$$

defined on the set of all (u, u') such that $\pi^\Lambda(q(u')) = \pi^\Lambda(s(u')) = p(u)$, is smooth as a map into the standard \mathbb{R} . It is sufficient to assume that U is connected and, subsequently, that U' is such that each of the plots q, s lifts to a plot of $(X_1 \cup_f X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$, moreover, one of form $(p, (q_1^\Omega, q_2^\Omega))$, where p lifts either to a plot p_1 of X_1 or a plot p_2 of X_2 ; notice that these two cases are perfectly analogous in the current context.

Suppose that p lifts to a plot p_1 of X_1 . The value of the corresponding evaluation map in this case is

$$\begin{aligned} g^\Lambda(p(u))(q(u'), s(u')) &= \\ &= \begin{cases} g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')), \tilde{\rho}_1^\Lambda(s(u'))) & \text{on } p_1^{-1}(X_1 \setminus Y) \\ \frac{1}{2} (g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')), \tilde{\rho}_1^\Lambda(s(u'))) + g_2^\Lambda(f(p_1(u)))(\tilde{\rho}_2^\Lambda(q(u')), \tilde{\rho}_2^\Lambda(s(u')))) & \text{on } p_1^{-1}(Y). \end{cases} \end{aligned}$$

It now suffices to apply the compatibility condition to the second part of the formula to obtain that

$$g^\Lambda(p(u))(q(u'), s(u')) = g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')), \tilde{\rho}_1^\Lambda(s(u')))$$

on the entire range of p , so it is smooth, because by assumption g_1^Λ is, in particular, smooth.

Likewise, if p lifts to a plot p_2 of X_2 then the corresponding evaluation has form

$$\begin{aligned} g^\Lambda(p(u))(q(u'), s(u')) &= \\ &= \begin{cases} g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')), \tilde{\rho}_2^\Lambda(s(u'))) & \text{on } p_2^{-1}(X_2 \setminus f(Y)) \\ \frac{1}{2} (g_1^\Lambda(f^{-1}(p_2(u)))(\tilde{\rho}_1^\Lambda(q(u')), \tilde{\rho}_1^\Lambda(s(u'))) + g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')), \tilde{\rho}_2^\Lambda(s(u')))) & \text{on } p_2^{-1}(f(Y)). \end{cases} \end{aligned}$$

In this case the compatibility condition ensures that

$$g^\Lambda(p(u))(q(u'), s(u')) = g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')), \tilde{\rho}_2^\Lambda(s(u'))),$$

and so we obtain a smooth function by the assumption on g^Λ . Both cases having thus been considered, we get the final claim. \square

A direct consequence of the theorem just proven is the following statement.

Corollary 9.4. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. If $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ admit compatible pseudo-metrics, then $\Lambda^1(X_1 \cup_f X_2)$ admits a pseudo-metric (induced by these given ones).*

References

- [1] J.D. CHRISTENSEN – G. SINNAMON – E. WU, *The D-topology for diffeological spaces*, arXiv.math/1302.2935v4.
- [2] J.D. CHRISTENSEN – E. WU, *Tangent spaces and tangent bundles for diffeological spaces*, arXiv:1411.5425v1.
- [3] P. IGLESIAS-ZEMMOUR, *Fibrations difféologiques et homotopie*, Thèse de doctorat d'État, Université de Provence, Marseille, 1985.
- [4] P. IGLESIAS-ZEMMOUR, *Diffeology*, Mathematical Surveys and Monographs, 185, AMS, Providence, 2013.
- [5] Y. KARSHON — J. WATTS, *Basic forms and orbit spaces: a diffeological approach*, SIGMA, 2016.
- [6] E. PERVOVA, *On the notion of scalar product for finite-dimensional diffeological vector spaces*, arXiv:1507.03787v1.
- [7] E. PERVOVA, *Diffeological vector pseudo-bundles*, Topology and Its Applications **202** (2016), pp. 269-300.
- [8] E. PERVOVA, *Diffeological vector pseudo-bundles, and diffeological pseudo-metrics on them as substitutes for Riemannian metrics*, arXiv:1601.00170v1.
- [9] E. PERVOVA, *Diffeological connections on diffeological vector pseudo-bundles*, arXiv:1611.07694v1.
- [10] J.M. SOURIAU, *Groups différentiels*, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Mathematics, 836, Springer, (1980), pp. 91-128.
- [11] J.M. SOURIAU, *Groups différentiels de physique mathématique*, South Rhone seminar on geometry, II (Lyon, 1984), Astérisque 1985, Numéro Hors Série, pp. 341-399.
- [12] A. STACEY, *Comparative smootheology*, Theory Appl. Categ., 25(4) (2011), pp. 64-117.
- [13] M. VINCENT, *Diffeological differential geometry*, Master Thesis, University of Copenhagen, 2008.
- [14] J. WATTS, *Diffeologies, Differential Spaces, and Symplectic Geometry*, PhD Thesis, 2012, University of Toronto, Canada.
- [15] E. WU, *Homological algebra for diffeological vector spaces*, arXiv:1406.6717v1.

University of Pisa
Department of Mathematics
Via F. Buonarroti 1C
56127 PISA – Italy

ekaterina.pervova@unipi.it